INT-PUB 04-09
UW/PT 04-04

A Viscosity Bound Conjecture

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March 2004

Abstract

Exploring an extension of the correspondence between black hole physics and thermodynamics to non-equilibrium processes, we show that the ratio of shear viscosity to volume density of entropy in theories with gravity duals is equal to a universal value of $\hbar/(4\pi)$. We conjecture that this value serves as a lower limit on the ratio of shear viscosity to entropy density for all systems realizable in nature.

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The discovery of Hawking radiation [1] confirmed that black holes are endowed with thermodynamic properties such as entropy and temperature, as first suggested by Bekenstein [2] and Bardeen, Carter and Hawking [3] based on the analogy between black hole physics and equilibrium thermodynamics. For black branes, i.e., black holes with translationally invariant horizons, thermodynamics can be extended to *hydrodynamics*—the theory that describes long-wavelength deviations from thermal equilibrium. Thus, black branes possess hydrodynamic properties of continuous fluids and can be characterized by kinetic coefficients such as viscosity, diffusion constants, etc. From the perspective of the holographic principle [4, 5], the hydrodynamic behavior of a black-brane horizon is identified with the hydrodynamic behavior of the dual theory.

In this Essay, we argue that in theories with gravity duals, the ratio of the shear viscosity to the volume density of entropy is equal to a universal value of $\hbar/4\pi$. Moreover, we argue that this is the lowest possible value for the ratio of shear viscosity to entropy density in all isotropic systems admitting hydrodynamic description.

In d spatial dimensions, shear viscosity η is measured in kg m^{2-d} s^{-1}. The volume density of entropy s is measured in m^{-d} (in units where Boltzmann constant is set to one). The ratio η/s thus has the dimension of kg m^2 s^{-1}, i.e., the same as the Planck constant $\hbar$. This observation is more than merely a curiosity, as we shall see shortly.

Consider a field theory dual to a black-brane metric. One can have in mind, as an example, the near-extremal D3 brane in type IIB supergravity,

$$ds^2 = \frac{r^2}{R^2}(-f dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{r^2 f} dr^2, \qquad f = 1 - \frac{r_0^4}{r^4}\,, \tag{1}$$

but our discussion is not tied to any specific form of the metric. All black branes have an event horizon ($r = r_0$ for the metric (1)), which is extended along several spatial dimensions (x, y, z in the case of (1)). The dual field theory is at a finite temperature, equal to the Hawking temperature of the black brane.

The entropy of the dual field theory is equal to the entropy of the black brane, which is proportional to the area of its event horizon,

$$S = \frac{A}{4G}\,, \tag{2}$$

where G is the Newton constant (we set $\hbar = c = 1$). For black branes A contains a trivial infinite factor V equal to the spatial volume along directions parallel to the horizon. The entropy density s is equal to $a/(4G)$, where $a = A/V$.

The shear viscosity of the dual theory can be computed from gravity in a number of equivalent approaches [6, 7, 8]. Here we use Kubo's formula, which relates viscosity to equilibrium correlation functions. In a rotationally invariant field theory,

$$\eta = \lim_{\omega\to 0} \frac{1}{2\omega} \int dt\, d\mathbf{x}\, e^{i\omega t} \left\langle [T_{xy}(t,\mathbf{x}),\, T_{xy}(0,\mathbf{0})] \right\rangle . \tag{3}$$

Here T_{xy} is the xy component of the stress-energy tensor (one can replace T_{xy} by any component of the traceless part of the stress tensor). We shall now relate the right hand side of (3) to the absorption cross section of low-energy gravitons.

According to the AdS/CFT correspondence [9], the stress-energy tensor $T_{\mu\nu}$ couples to metric perturbations at the boundary. Following Klebanov [10], let us consider a graviton of frequency ω, polarized in the xy direction, and propagating perpendicularly to the brane. In the field theory picture, the absorption cross section of the graviton by the brane measures the imaginary part of the retarded Greens function of the operator coupled to h_{xy}, i.e., T_{xy},

$$\sigma_{\rm abs}(\omega) = -\frac{2\kappa^2}{\omega}\,{\rm Im}\,G^{\rm R}(\omega) = \frac{\kappa^2}{\omega}\int dt\,d\mathbf{x}\,e^{i\omega t}\,\langle [T_{xy}(t,\mathbf{x}),\,T_{xy}(0,\mathbf{0})]\rangle\,, \tag{4}$$

where $\kappa = \sqrt{8\pi G}$ appears due the normalization of the graviton's action. Comparing (3) and (4), one finds

$$\eta = \frac{\sigma_{\rm abs}(0)}{2\kappa^2} = \frac{\sigma_{\rm abs}(0)}{16\pi G}\,. \tag{5}$$

The absorption cross section $\sigma_{\rm abs}$, on the other hand, is calculable classically by solving the linearized wave equation for h^x_y. It can be shown (see Appendix) that under rather general assumptions the equation for h^x_y is the same as that of a minimally coupled scalar. The absorption cross section for the scalar is constrained by a theorem [11, 12], which states that in the low-frequency limit $\omega \to 0$ this cross section is equal to the area of the horizon, $\sigma_{\rm abs} = a$. Since $s = a/4G$, one immediately finds that

$$\frac{\eta}{s} = \frac{\hbar}{4\pi}\,, \tag{6}$$

where $\hbar$ is now restored. This shows that the ratio η/s does not depend on the concrete form of the metric within the assumptions of [11, 12]. Indeed, this ratio is the same for Dp [6, 8], M2 and M5 [13] branes and deformations of the D3 metric [8, 14].

Dual gravity description of gauge theories is valid in the regime of infinitely strong coupling. As Eq. (6) shows, in this regime the ratio η/s appears to be universal (independent of the coupling constant and other microscopic details of the theory). On the other hand, from perturbative calculations in finite-temperature field theory it is known that the ratio η/s approaches infinity in the limit of vanishing coupling.

This observation prompts us to formulate the "viscosity bound" conjecture: in all isotropic liquids, gases and plasmas realizable in nature, the ratio of shear viscosity to entropy density cannot be smaller than the value of this ratio in theories with gravity duals:

$$\frac{\eta}{s} \geqslant \frac{\hbar}{4\pi}\,. \tag{7}$$

First let us check the conjecture for the most ubiquitous fluid—water. Under normal conditions (P = 0.1 MPa, T = 298.15 K) the viscosity of water is $\eta \approx 0.89 \times 10^{-3}$ Pa s

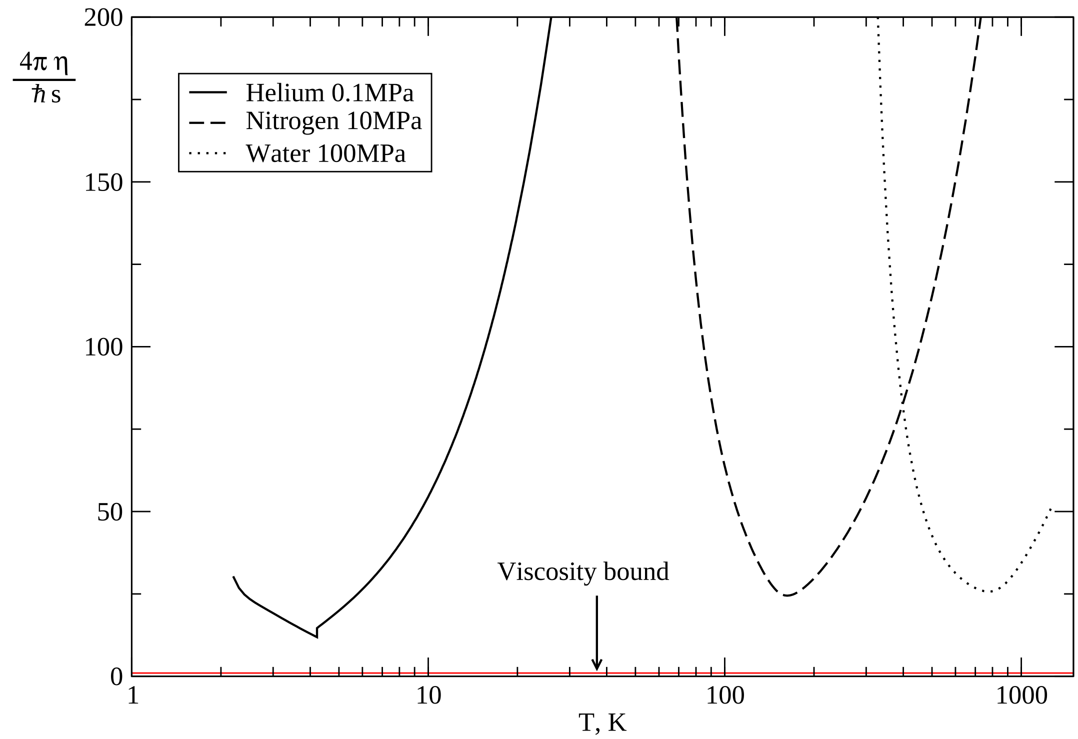

Figure 1: The viscosity-entropy ratio for some common substances.

and the entropy density is $s \approx 2.8 \times 10^{29}$ m^{-3}. The ratio η/s is 380 times larger than $\hbar/(4\pi)$. Using standard tables [15] one can find η/s for many liquids and gases at different temperatures and pressures. Figure 1 shows temperature dependence of η/s, normalized by $\hbar/(4\pi)$, for a few substances at different pressures. It is clear that the viscosity bound is well satisfied for these substances. Liquid helium reaches the smallest value of η/s, but this value still exceeds the bound by a factor of about 10.

It is important to avoid some common misconceptions which at first sight seem to invalidate the viscosity bound. One might think that an ideal gas has an arbitrarily small viscosity, which violates the bound. However, the viscosity of a gas *diverges* when the interaction between molecules is turned off. This is because for gases viscosity is proportional to the mean free path of the molecules. The second common misconception involves superfluids, which seem to have zero viscosity. However, according to Landau's two-component theory, superfluids have finite and measurable shear viscosity associated with the normal component.

It is useful to compare the viscosity bound with two other bounds widely discussed in the literature: the entropy bound (e.g., in its covariant formulation [16]) which states that the entropy of a region of space is limited by the area A of the region's boundary,

$$S \leqslant \frac{c^3}{\hbar G}\frac{A}{4}\,, \tag{8}$$

and the Bekenstein bound [17], which states that the entropy of a system is limited by the product of its linear size R and mass M,

$$S \leqslant \frac{c}{\hbar} 2\pi RM\,. \tag{9}$$

The viscosity bound can also be written as an upper bound on the entropy,

$$S \leqslant \frac{4\pi}{\hbar}\eta V\,, \tag{10}$$

where V is the total volume. It is similar to the Bekenstein bound in the sense that it covers non-gravitating systems despite having the origin in the theory of gravitation. The viscosity bound stands apart as the only bound that does not involve the speed of light. Because of this feature, it is relevant for non-relativistic systems, in contrast to the other bounds. One may hope that it is possible to relate the viscosity bound to the two other bounds, and eventually to the generalized second law of thermodynamics. It will be important to learn whether the viscosity bound suffers from the same problems as the other bounds such as the species problem.

The bound (7) is most useful for strongly interacting systems where no theoretical calculation of viscosity is possible. One of such systems is the quark-gluon plasma (QGP) created in heavy ion collisions which behaves in many respects as a droplet of a liquid. As the viscosity of the QGP is not computable reliably except at inaccessibly high temperatures, the information provided by the conjectured bound is extremely valuable. There are experimental hints that the viscosity of the QGP at temperatures achieved by the Relativistic Heavy Ion Collider is surprisingly small, possibly close to saturating the viscosity bound [18]. Further investigations may reveal whether the QGP conforms to the viscosity bound.

Another possible application of the viscosity bound is trapped atomic gases. By using the Feshbach resonance, strongly interacting Fermi gases of atoms have been created recently. These gases have been observed to behave hydrodynamically [19]. Currently the viscosity bound is the only source of information about the viscosity of these gases. It would be very interesting to test the bound experimentally using these gases.

This work is supported by DOE grants and the Alfred P. Sloan Foundation.

Appendix

In this Appendix, we elaborate on the statement made after Eq. (5) that under rather general assumptions the equation for h_y^x is identical to the one obeyed by a minimally coupled massless scalar.

Consider a D-dimensional solution to Einstein's equations of the form

$$ds^2 = g^{(0)}_{MN} dx^M dx^N = f(z)\,(dx^2 + dy^2) + g_{\mu\nu}(z)\, dz^\mu dz^\nu\,. \tag{11}$$

We shall be interested in small perturbations around the metric, $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$. We assume that the only non-vanishing component of $h_{\mu\nu}$ is h_{xy}, and that it does not depend on x and y: $h_{xy} = h_{xy}(z)$. This field has spin 2 under the O(2) rotational symmetry in the xy plane, which implies that all other components of $h_{\mu\nu}$ can be consistently set to zero [7]. We now show that $h^x_y = h_{xy}/f$ obeys the equation for a minimally coupled massless scalar in the background (11).

Einstein's equations can be written in the form

$$R_{MN} = T_{MN} - \frac{T}{D-2}\, g_{MN}\,, \tag{12}$$

where the stress-energy tensor T_{MN} and its trace T depend on other fields such as the dilaton and various forms supporting the background (11) (for example, the fields appearing in the low energy type II string theory). Again, O(2) xy rotational symmetry implies that all perturbations of matter fields can be set to zero consistently. Thus when M and N are x or y, the right hand side of Einstein's equations reads

$$T_{\alpha\beta} - \frac{T}{D-2}\, g_{\alpha\beta} = -\left(\mathcal{L} + \frac{T^{(0)}}{D-2}\right)(\delta_{\alpha\beta} f + h_{\alpha\beta}), \qquad \alpha, \beta = x, y, \tag{13}$$

where $\mathcal{L}$ is the Lagrange density of matter fields and $T^{(0)}$ is the trace of the unperturbed stress-energy tensor. Substituting the unperturbed metric (11) into Einstein's equations, one finds

$$\frac{1}{2}\left[\frac{\Box f}{f} - \frac{(\partial f)^2}{f^2}\right] = \mathcal{L} + \frac{T^{(0)}}{D-2}\,. \tag{14}$$

Expanding Einstein's equations to linear order in h_{xy}, one has

$$R_{xy} = -\frac{1}{2}\Box h_{xy} + \frac{1}{f}\partial^\mu f\, \partial_\mu h_{xy} - \frac{(\partial f)^2}{2f^2} h_{xy} = -\left(\mathcal{L} + \frac{T^{(0)}}{D-2}\right) h_{xy}\,. \tag{15}$$

Combining Eqs. (14) and (15), we obtain an equation for h_{xy}

$$\Box\, h_{xy} - 2\,\frac{\partial^\mu f}{f}\, \partial_\mu h_{xy} + 2\,\frac{(\partial f)^2}{f^2}\, h_{xy} - \frac{\Box f}{f}\, h_{xy} = 0\,. \tag{16}$$

Changing the variable from h_{xy} to $h^x_y = h_{xy}/f$, one can see that h^x_y indeed satisfies the equation for a minimally coupled massless scalar: $\Box h^x_y = 0$.

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