Notes for Quantum Mechanics

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Lecture 1
From the 19th to the 20th Century

This is a set of notes following the book starting in chapter 2. Its a general introduction to the ideas.

Nineteenth Century Physics - Mechanics and E and M, particles and waves

Early in the 20th century, there was Newton's understanding of mechanics, and Maxwell's understanding of E and M. For mechanics we start with some quantities, $\vec{r}$, $\vec{v}$, $\vec{p}$, E and some relationships $\vec{F} = m\vec{a}$, $E = \frac{p^2}{2m} + V(r)$ and for gravity - a force law: $F = \frac{\text{GMm}}{r^2}$. These explained the motion of objects from balls on earth to the orbits of planets- that is particles.. Also early in the century, Einstein arrived at his special theory of relativity - that is his famous $E = \text{mc}^2$.This is for a particle at rest. For a moving particle

$$\text{E} = \sqrt{\text{p}^2\,\text{c}^2 + \text{m}^2\,\text{c}^4} \quad \text{this is the relativistic formula. We will use it later.} \tag{1}$$

Within the context of mechanics, we also had mechanical waves whose wave eqn can be written (in 1-D)

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 u(x,t)}{\partial t^2}$$

This and its 3-D cousin governed mechanical waves - e.g. waves on a string, ocean waves, sound etc. If you thought about it, you might think (and you would be right) that you wouldn't need these equations since air, water and strings are just lots of little particles hooked together. If you had a big enough computer, you could just use F=ma and do everything you needed.

For Electricity and Magnetism, we have in integral form which you have seen in your intro physics classes.

$$\oint \vec{E}\cdot d\,\vec{A} = \frac{q}{\epsilon_0} \;-\; \text{Gauss's Law - this just says there is an electric field and charges give an electric field} \tag{2}$$

$$\oint \vec{B}\cdot d\,\vec{A} = 0 \;-\; \text{Gauss's Law for magnetism - this says there is a magnetic field - there are no magnetic charges} \tag{3}$$

$$\oint \vec{E}\cdot d\,\vec{s} = -\frac{\text{d}\Phi_B}{\text{dt}} \;-\; \text{Electric fields can be made by changing magnetic flux} \tag{4}$$

$$\oint \vec{B}\cdot d\,\vec{s} = \mu_0\, i + \frac{1}{c^2}\,\frac{\partial}{\partial t}\int \vec{E}\cdot d\,\vec{A} \;-\; \text{B Fields can be made by a current, AND by the displacement current} \tag{5}$$

Eqn's 2-5 are the Maxwell eqns in integral form. These unified electricity and magnetism. The two seemingly different forces - electric forces, and magnetic forces are really different manifestations of the same phenomena - the Electromagnetic Force. There is also another amazing thing which is hinted at when one looks at the second term on the

RHS - the so-called displacement current. If you remember - this can most easily be seen if one considers a capacitor in a circuit. Just after the circuit is closed, a "fake current" goes through the capacitor (of course no real charge goes through) which in fact can make a magnetic field. This means that the E and B fields would be able to see-saw off each other with an E field making a B field, and then the B field making and E field etc without any charges (or sources) around! These fields could have a ghostly existence free of any source. One can see this if one combines eqn's 3 and 4 into the wave eqns.

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2} \tag{6}$$

This means that the Maxwell eqn's predict a wave made of E and B fields. What is this?? - of course it's just light - electromagnetic waves. Light can move through empty space with no charges moving around. So arising out of our attempt to understand E and B fields - studied in current loops etc, we get an explanation of light. The solutions of these equations is a wave which go like sines and cosines. What do we know about waves? Well if we consider waves in a bathtub, we realize that we should get the phenomena of interference. When you drop two stones in still pool of water, you get a nice pattern which arises because the waves interfere with each other constructively and destructively. We can test this. We use a laser and split it into two pieces by using a double slit. If you look at the result by projecting against a white wall, one sees, not two spots that we might expect if light were like little bullets, but rather a nice interference pattern expected from waves - just like the pattern from dropping stones into a pool. But wait a minute!. What is it that waves? For sound we have air molecules - for ocean waves - we have water molecules. What waves for EM waves? It was shown by the Michelson-Morely experiment that there was nothing which waved - i.e. no ether. So now we really do have real "waves" as opposed to particles. You cannot get a light wave by using F=ma!

Nineteenth Century Physics in trouble: The photoelectric effect and the spectrum of hydrogen - and some crazy ideas

Note the situation here. We have a set of equations which have to do with particles, and another set which described waves. The world spits into two pieces. Can these two realms talk to each other?

The unification of Electricity on the one hand, and Magnetism on the other - to predict light waves - a "fundamental thing" was an enormous triumph. Maxwell thought that physics would be over in a few years. How wrong he was! There were a few problems. The classic thing which bothered folks was black body radiation, however I would like to focus on two other phenomena. The first is the spectrum of hydrogen, the second is the photo-electric effect. Let's start with the photo-electric effect. It was known that light waves could make an electric charge move. If one shined light on a metal - electrons would be ejected. The problem here is that the speed of the electron coming out had no dependence on the intensity of light. There is a potential called a work function which holds the electron and keeps it from getting ejected. If one shined light at a metal - and no electrons came out - one would assume that you could just turn up the brightness and then finally the electrons would be ejected. This was NOT true. It did not depend on the intensity of light. However if one made the wavelength of light smaller, even if it was very dim, electrons would begin to be ejected. The smaller the wavelength (i.e. larger the frequency) the faster the electron would emerge. Einstein proposed that the energy of light could be written as

$E=h\nu$ where h is Plank's constant - for the moment DEFINED by this equation. (7)

Now lets mix-up waves and particles. Light is a wave - but for the moment, lets assume that its a particle(this is what Newton thought), and its energy is hν. Now we will use Einstein's equation (1). What is the mass? Well since light travels at the speed of light - this implies that the mass must be zero. So eqn 1 becomes E=pc. So we can set pc=hν and solve for p.

$$p = \frac{h}{c}\,\nu = \frac{h}{\lambda} \qquad (8)$$

Now since we are pretending that light is a particle, we can consider the process of ejecting a photon as the collision of a light particle and an electron. Clearly then the larger the frequency - the larger the momentum, and the more velocity will be imparted to the electron. So is light a particle or a wave??? [I won't answer this here] Ironically even though Einstein was the father of the photo-electric effect (for which he got a Nobel prize) and relativity - eqn 1, which is the foundation for this first idea that maybe light is like a particle, even though its a wave - he never really believed in QM.

We have now pretended a wave was a particle. Lets turn to the spectrum of hydrogen. It was known that moving charges caused EM waves - light . A fluorescent lamp makes light when a current is sent through a gas and ionizes it. When electrons recombine with the ionized gas, a light wave (photon) is produced.There was a problem though. It would seem that the electron should be able to end up at any radius around the central proton (lets think about hydrogen now), just like there can be planets at any radius around the sun. When one looks at the spectrum from hydrogen, however, it has spikes. That is - only certain wavelengths are produced. It is as if the electrons can fall into orbits at certain fixed radii. It was also known that the ground state, that is the innermost orbital had an energy of -13.6 eV. This is weird. Its as if you had a merry-go-round that would only go a 1mph, 5mph, and 10mph, and NOTHING in between. Bohr said - lets pretend that electrons are waves. We imagine that this wave goes around in a circle around the proton. If the circumference is not an integral number of wavelengths - the electron-wave will go around and interfere destructively and kill itself off. This means that only radii which give circumferences which are integer multiples of the wavelength λ will be allowed orbits - thereby solving the problem. Note that now, we have assumed a particle was a wave.

Let's now see if we can't naively use some of these ideas, and calculate the allowed energy shells of hydrogen - from the standard equations we have listed above. First if we look at the electron going around the proton, the orbit will be such that the total force radially on the electron is zero [using F=ma] $\frac{\mathrm{mv}^2}{r} = \frac{e^2}{r^2}$

$$\mathrm{mv}^2 = \frac{e^2}{r} \qquad (9)$$

We will also use this to solve for the angular momentum which we use later. $L^2 = m^2\, v^2\, r^2 = \mathrm{me}^2\mathrm{r}$

Now we like to use Bohr's idea-i.e. that an electron is a wave. $2\pi\mathrm{r}=\mathrm{n}\lambda$ where n is an integer > 1 for constructive interference, so $\mathrm{r}=\frac{\mathrm{n}\lambda}{2\pi}$

In addition we will use the idea that a wavelength can be converted into a momentum. We get this by using Einstein's eqn (1) for a "photon" setting the mass equal to zero so E=pc.

We then use another equation of Mr. Einstein (7) getting E=pc=hν. Now if you remember $c=\lambda\nu$ so pc = $\frac{\mathrm{hc}}{\lambda}$ and finally $\lambda = \frac{h}{p}$

So we get for the radius $\mathrm{r}=\frac{\mathrm{n}\lambda}{2\pi}=\frac{n}{2\pi}\,\frac{h}{p} = \frac{\mathrm{n}\hbar}{\mathrm{mv}}$ where we have used p=mv. [this is OK. if you figure out the velocity of the electron, its about 1% the speed of light]. If we now find the angular momentum we get L=mvr=n$\hbar$ This is now the quantization of angular momentum which we will see later. Now using (9) we have $L^2=n^2 \hbar^2 = \text{me}^2\text{r}$. After solving for r

$$r=\frac{n^2 \hbar^2}{\text{me}^2} \tag{10}$$

Now we want to find the energy, and using (9) and then (10)

$$E = \frac{1}{2}\,\text{mv}^2 - \frac{e^2}{r} = \frac{1}{2}\frac{e^2}{r} - \frac{e^2}{r} = -\frac{e^2}{2r} = -\frac{e^2\, m\, e^2}{2 n^2 \hbar^2} = -\frac{\text{me}^4}{2\hbar^2}\,\frac{1}{n^2}$$

now the combination $\mathcal{R}=\frac{\text{me}^4}{2\hbar^2}$ =13.6 eV, the Rydberg constant.

So the energy of the lowest orbital (n=1) is what was known experimentally -13.6 eV. So there is something to this crazy calculation. Next if electrons can be considered waves, then they better show interference effects - just like light. Davison and Germer did this and in fact were able to see electron diffraction JUST LIKE light.

It is important to keep a rule in mind

When testing a theory - the crucial question is - does it match to experiment. If not - we throw the theory out. If so, we have to take the theory - no matter how crazy, seriously.

So it is clear that we need to have an entirely new way to think about things. Somehow waves are like particles [the photo-electric effect] and particles are like waves [the Bohr model] So - to the birth of Quantum Mechanics.

Some easy things to remember

$1\text{ fm} = 10^{-15}\ m \quad (1\text{ femto} - \text{meter, or } 1\text{ fermi})$

$\hbar$c=197 MeV-fm (11)

$\alpha=\frac{e^2}{\hbar c} = \frac{1}{137}$

$m_e\, c^2 = 0.5\,\text{MeV}/c^2$

We can use this to figure out the Rydberg constant

$$\mathcal{R} = \frac{\text{me}^4}{2\hbar^2} = \frac{\text{mc}^2\, e^4}{2\hbar^2\, c^2} = \frac{\text{mc}^2\, \alpha^2}{2} = \frac{1}{2}0.5\text{ MeV}\left(\frac{1}{137}\right)^2 = 13.3\text{ MeV (not bad)}$$

how about the velocity to check if p=mv is OK to use.

L=mvr=n$\hbar$ $\quad v = \frac{\hbar}{\text{mr}} = \frac{\hbar}{m}\left(\frac{\text{me}^2}{\hbar^2}\right) = \frac{e^2}{\hbar} \quad \frac{v}{c} = \frac{e^2}{\hbar c} = \frac{1}{137}$ or about 1% the speed of light.

Using these tricks makes (some) numerical calculations easy.

