Physics 156A Quantum Mechanics - Midterm

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Soulutions

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{2004, 10, 30, 21, 41, 28.9433664}
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Total - 70 pts
pr 1 - 25 pts
pr 2 - 20 pts
pr 3 - 25 pts
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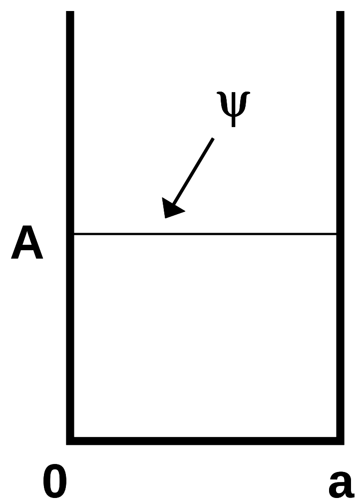
1) You have a wave function for a particle in a box with sides at 0 and a. The wave function looks like ψ(x)=A - that is - its just a constant.



a) what is the normalization constant, A?

$\int_0^a A^2 \, dx = A^2 \, x \,|^a{}_0 = A^2 \, a = 1 \qquad A = \frac{1}{\sqrt{a}}$ so $\psi(x) = \frac{1}{\sqrt{a}}$

b) what is the expectation value of x, that is $\langle \hat{x} \rangle$?

$\langle \hat{x} \rangle = \langle \psi \mid \hat{x} \mid \psi \rangle = \int_0^a \frac{1}{\sqrt{a}} \, x \, \frac{1}{\sqrt{a}} \, dx = \frac{1}{a} \int_0^a \mathrm{xdx} = \frac{1}{a} \, \frac{x^2}{2} \, \{x, 0, a\} = \frac{1}{a} \, \frac{a^2}{2} = \frac{a}{2}$ - makes sense

c) what are the eigenstates and associated energies for a particle in a box such as the one you are looking at?

for the particle in a box its $\varphi_n(x) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{n\pi x}{a}\right)$ with energies $E_n = n^2 E_1$ where $E_1 = \frac{\hbar^2 \pi^2}{2\, m a^2}$

d) do an expansion of ψ in terms of these eigenstates

$\psi(x)=\sum_{i=0}^{\infty} c_n \varphi_n$ where $c_n=\int_0^a \varphi_n(x)\psi(x)dx$

$c_n = \int_0^a \varphi_n(x)\psi(x)dx = \frac{1}{\sqrt{a}} \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi x}{a}\right) dx = \frac{\sqrt{2}}{a} \left(\frac{-a}{n\pi}\right) \left[\cos\left(\frac{n\pi x}{a}\right)\right] \{x, 0, a\} = \frac{-\sqrt{2}}{n\pi} [\cos(n\pi)-1]=$

$= \frac{\sqrt{2}}{n\pi} [1-\cos(n\pi)] = \frac{\sqrt{2}}{n\pi} [1-(-1)^n]$

e) what is the probability of finding the lowest energy? (Often called the ground state)

probability$(n = 1) = \left(\frac{2\sqrt{2}}{\pi}\right)^2 \sim \frac{8}{10} \sim .8$ its more like .81

For your information:

the n=2 state has probability zero

n=3 has $\left(\frac{2\sqrt{2}}{3\pi}\right)^2 \sim .09$

n=4 has zero

n=5 has $\left(\frac{2\sqrt{2}}{5\pi}\right)^2 \sim .03$

2) You begin with a mixture of $\frac{2}{3}\ |S_x; +\rangle$ (by probability) and $\frac{1}{3}|S_x; -\rangle$. Correctly normalized this would be a state $|\alpha\rangle=\sqrt{\frac{2}{3}}\ |S_x; +\rangle+\sqrt{\frac{1}{3}}\ |S_x; -\rangle$.

a) Find $\langle \hat{S}_x \rangle$

We know $\hat{S}_x \langle S_x; + | = \frac{\hbar}{2}|S_x; +\rangle$ and $\hat{S}_x |S_x; -\rangle = -\frac{\hbar}{2}|S_x; -\rangle$

$\langle \hat{S}_x \rangle = (\sqrt{\frac{2}{3}} \langle S_x; +|+\sqrt{\frac{1}{3}} \langle S_x; -|)\hat{S}_x(\sqrt{\frac{2}{3}} |S_x; +\rangle+\sqrt{\frac{1}{3}} |S_x; -\rangle.\ )=\frac{2}{3}\langle S_x; +|\hat{S}_x|S_x; +\rangle+\frac{1}{3}\langle S_x; -|\hat{S}_x|S_x; -\rangle=$

$=\frac{2}{3}\ \frac{\hbar}{2}\langle S_x; +|S_x; +\rangle-\frac{1}{3}\ \frac{\hbar}{2}\langle S_x; -|S_x; -\rangle=\frac{\hbar}{2}\left(\frac{2}{3}-\frac{1}{3}\right)=\frac{\hbar}{6}$

b) Find $\langle \hat{S}_z \rangle$

This is a bit more difficult. We write

$| S_x; +\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$ and $| S_x; -\rangle = \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle)$

$|\alpha\rangle=\sqrt{\frac{2}{3}}\ \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)+\sqrt{\frac{1}{3}}\ \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle)=[(\sqrt{\frac{1}{3}} +\sqrt{\frac{1}{6}})|+\rangle-(\sqrt{\frac{1}{3}} -\sqrt{\frac{1}{6}})|-\rangle]$

So $\langle \hat{S}_z \rangle=[(\sqrt{\frac{1}{3}} +\sqrt{\frac{1}{6}})\langle+|-(\sqrt{\frac{1}{3}} -\sqrt{\frac{1}{6}})\langle-|]S_z[(\sqrt{\frac{1}{3}} +\sqrt{\frac{1}{6}})|+\rangle-(\sqrt{\frac{1}{3}} -\sqrt{\frac{1}{6}})|-\rangle]=$

$= \left(\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{6}}\right)\left(\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{6}}\right)\langle + | S_z |+\rangle + \left(\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{6}}\right)\left(\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{6}}\right)\langle - | S_z |-\rangle =$

$\frac{\hbar}{2}\left[\left(\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{6}}\right)\left(\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{6}}\right) - \left(\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{6}}\right)\left(\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{6}}\right)\right] =$

$\frac{\hbar}{2}\left[\left(\frac{1}{3} + \frac{1}{6} + 2\sqrt{\frac{1}{18}}\right) - \left(\frac{1}{3} + \frac{1}{6} - 2\sqrt{\frac{1}{18}}\right)\right] = \frac{\hbar}{2}\, 4\sqrt{\frac{1}{18}} = \frac{\hbar}{2}\, 4\, \frac{1}{3}\sqrt{\frac{1}{2}} = \hbar\, \frac{\sqrt{2}}{3} \sim 0.47\,\hbar$

3) You have an operator $\hat{U}=(\hat{S}_x{}^2+\hat{S}_y{}^2 + \hat{S}_z{}^2) =(\hat{S}_x\hat{S}_x+\hat{S}_y\, \hat{S}_y + \hat{S}_z\, \hat{S}_z)$ [for those thinking ahead $\hat{U}$ will turn out to be an operator we call $\vec{\hat{S}}^2$]

a) Is $\hat{U}$ hermitian? Show it if it is. Prove that it is not, if it is not.

We need to show that $\hat{U}^{\dagger}=\hat{U}$

Use $(\hat{X}\,\hat{Y})^{\dagger}=\hat{Y}^{\dagger}\,\hat{X}^{\dagger}$ We also know that $\hat{S}_x$,$\hat{S}_y$ and $\hat{S}_z$ are hermitian

$\hat{U}^{\dagger}=(\hat{S}_x\hat{S}_x+\hat{S}_y\,\hat{S}_y+\hat{S}_z\,\hat{S}_z)^{\dagger}=(\hat{S}_x^{\dagger}\hat{S}_x^{\dagger}+\hat{S}_y^{\dagger}\,\hat{S}_y^{\dagger}+\hat{S}_z^{\dagger}\,\hat{S}_z^{\dagger})=(\hat{S}_x\hat{S}_x+\hat{S}_y\,\hat{S}_y+\hat{S}_z\,\hat{S}_z)=\hat{U}$

So it is hermitian

b) Show that $|+\rangle$ and $|-\rangle$ eigenkets of $\hat{U}$. Also give the eigenvalues

There are several ways of doing this You can do it in matrix notation. I will show one which uses the bra's and ket's

Remember that

$\hat{S}_x=\frac{1}{2}(\hat{S}_+ + \hat{S}_-) = \frac{\hbar}{2}(\,|+\rangle\langle-| + |-\rangle\langle+|\,)$ so $\hat{S}_x^{\,2}=\hat{S}_x\,\hat{S}_x=\frac{\hbar^2}{4}(\,|+\rangle\langle-| + |-\rangle\langle+|\,(\,|+\rangle\langle-| + |-\rangle\langle+| = \frac{\hbar^2}{4}(\,|+\rangle\langle+| + |-\rangle\langle-|\,)$

$\hat{S}_y=\frac{1}{2i}(\hat{S}_+ - \hat{S}_-)=\frac{\hbar}{2i}(\,|+\rangle\langle-| - |-\rangle\langle+|\,)$ so $\hat{S}_y^{\,2}=\hat{S}_y\,\hat{S}_y=-\frac{\hbar^2}{4}(\,|+\rangle\langle-| - |-\rangle\langle+|\,(\,|+\rangle\langle-| - |-\rangle\langle+| =$

$=-\frac{\hbar^2}{4}(\,-|+\rangle\langle+| - |-\rangle\langle-|\,)=\frac{\hbar^2}{4}(\,|+\rangle\langle+| + |-\rangle\langle-|\,)$

$\hat{S}_z=\frac{\hbar}{2}(\,|+\rangle\langle+| + |-\rangle\langle-|\,)$ so $\hat{S}_z^{\,2}=\hat{S}_z\,\hat{S}_z=\frac{\hbar^2}{4}(\,|+\rangle\langle+| + |-\rangle\langle-|\,(\,|+\rangle\langle+| + |-\rangle\langle-| = \frac{\hbar^2}{4}(\,|+\rangle\langle+| + |-\rangle\langle-|\,)$

then $\hat{U}=(\hat{S}_x^{\,2}+\hat{S}_y^{\,2}+\hat{S}_z^{\,2}) = 3\frac{\hbar^2}{4}(\,|+\rangle\langle+| + |-\rangle\langle-|\,)$

$\hat{U}|+\rangle=\frac{3\hbar^2}{4}(\,|+\rangle\langle+| + |-\rangle\langle-|\,)|+\rangle=\frac{3\hbar^2}{4}\,|+\rangle\langle+|+\rangle=\frac{3\hbar^2}{4}\,|+\rangle$ so $|+\rangle$ is a ket with eigenvalue $\frac{3\hbar^2}{4}$

$(\,|+\rangle\langle+| + |-\rangle\langle-|\,)|-\rangle=\frac{3\hbar^2}{4}(\,|+\rangle\langle+| + |-\rangle\langle-|\,)|-\rangle=\frac{3\hbar^2}{4}\,|-\rangle\langle-|-\rangle=\frac{3\hbar^2}{4}\,|-\rangle$ so $|-\rangle$ is a ket with eigenvalue $\frac{3\hbar^2}{4}$ i.e. the same as for $|+\rangle$

c) Is $|S_x;+\rangle$ an eigenket of $\hat{U}$ - (prove it, and if it is- give the eigenvalue)

$|\,S_x;+\rangle=\frac{1}{\sqrt{2}}(\,|+\rangle + |-\rangle\,)$

$\hat{U}\,|\,S_x;+\rangle=3\frac{\hbar^2}{4}(\,|+\rangle\langle+| + |-\rangle\langle-|\,)\frac{1}{\sqrt{2}}(\,|+\rangle + |-\rangle\,) = 3\frac{\hbar^2}{4}\,\frac{1}{\sqrt{2}}(\,|+\rangle + |-\rangle) = 3\frac{\hbar^2}{4}\,|\,S_x;+\rangle$

So $|\,S_x;+\rangle$ is also an eigenket with an eigenvalue $\frac{3\hbar^2}{4}$