Physics 156B Quantum Mechanics - Midterm

February 15, 2005

Solutions

Date[]

{2005, 2, 14, 20, 8, 39.6652784}

Total - 60 pts
pr 1 - 20 pts
pr 2 - 20 pts
pr 3 - 20 pts

1) You are given an operator $\hat{A}$ representing some observable which commutes with the hamiltonian, i.e. $[\hat{H},\hat{A}]=0$.

a) What is the relationship between the eigenstates of $\hat{H}$ and $\hat{A}$?

They share the same eigenstates. [additional info - the eigenvalues of one may be degenerate, and we often use the other to differentiate them. This will be the case for J and J_z]

b) Suppose the properly normalized eigenstates of $\hat{A}$ are $|a_i\rangle$. What is the value if $\langle a_i | a_j \rangle$?

$\langle a_i | a_j \rangle = \delta_{ij}$ That is they form a orthonormal set of base kets which span the space

c) What can you say about the eigenvalues a_i?

Since $\hat{A}$ is an observable, the a_i are real.

d) What can you say about the time dependence of $\langle \hat{A} \rangle$?

It is time independent since $\frac{d\langle \hat{A} \rangle}{dt} = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}] \rangle = 0$

e) Suppose you wanted to solve some sort of problem involving $\hat{H}$ and $\hat{A}$. What is the general solution you would write down? (There may be lots of unknown constants - that is OK)

$|\alpha\rangle = \sum c_i |a_i\rangle$

2) Consider a particle subject to a one dimensional simple harmonic oscillator potential $\frac{1}{2}\,m\omega^2\,\hat{x}^2$. It is in a mixture of n=0 and n=1 states, i.e. $|\alpha\rangle=\frac{1}{\sqrt{2}}\,(|0\rangle+|1\rangle)$. Find

a) $\langle\hat{x}\rangle$

$\hat{x}=\sqrt{\frac{\hbar}{2\,m\omega}}\,(\hat{a}^\dagger+\hat{a})\quad |1\rangle=\hat{a}^\dagger\,|0\rangle\quad E_1=\frac{3}{2}\,\hbar\omega\;E_0=\frac{1}{2}\hbar\omega\qquad \hat{a}^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle$ and $\hat{a}|n\rangle=\sqrt{n}\,|n-1\rangle$

$\langle\hat{x}\rangle=\frac{1}{2}\sqrt{\frac{\hbar}{2\,m\omega}}\,(\langle 0|+\langle 1|)(\hat{a}^\dagger+\hat{a})\,(|0\rangle+|1\rangle)=\frac{1}{2}\sqrt{\frac{\hbar}{2\,m\omega}}\,(\langle 1|\hat{a}^\dagger|0\rangle+\langle 0|\hat{a}|1\rangle)=\frac{1}{2}\sqrt{\frac{\hbar}{2\,m\omega}}\,(\langle 1|1\rangle+\langle 0|0\rangle)=\sqrt{\frac{\hbar}{2\,m\omega}}$

b) $\langle\hat{p}\rangle$

$\hat{p}=i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^\dagger-\hat{a})$

$\langle\hat{p}\rangle==\frac{1}{2}i\sqrt{\frac{m\hbar\omega}{2}}\,(\langle 0|+\langle 1|)(\hat{a}^\dagger-\hat{a})\,(|0\rangle+|1\rangle)=\frac{1}{2}i\sqrt{\frac{m\hbar\omega}{2}}\,(\langle 1|\hat{a}^\dagger|0\rangle\text{-}\langle 0|\hat{a}|1\rangle)=\frac{1}{2}i\sqrt{\frac{m\hbar\omega}{2}}\,(\langle 1|1\rangle\text{-}\langle 0|0\rangle)=0$

So at t=0, its an one end of its oscillations.

c) $\langle\hat{x}^2\rangle$

We know $[\hat{a},\hat{a}^\dagger]=1$ so $\hat{a}\,\hat{a}^\dagger=1+\hat{a}^\dagger\,\hat{a}$

$\hat{x}^2=\frac{\hbar}{2\,m\omega}(\hat{a}^\dagger+\hat{a})(\hat{a}^\dagger+\hat{a})=\frac{\hbar}{2\,m\omega}(\,\hat{a}^\dagger\hat{a}^\dagger+\hat{a}^\dagger\,\hat{a}+\hat{a}\,\hat{a}^\dagger+\hat{a}\,\hat{a}\,)=\frac{\hbar}{2\,m\omega}(\,\hat{a}^\dagger\hat{a}^\dagger+\hat{a}^\dagger\,\hat{a}+1+\hat{a}^\dagger\,\hat{a}+\hat{a}\,\hat{a}\,)$

$=\frac{\hbar}{2\,m\omega}(\hat{a}^\dagger\hat{a}^\dagger+2\,\hat{a}^\dagger\,\hat{a}+1+\hat{a}\,\hat{a}\,)$

$\langle\hat{x}^2\rangle=\frac{1}{2}\,\frac{\hbar}{2\,m\omega}((\langle 0|+\langle 1|)(\,\hat{a}^\dagger\,\hat{a}^\dagger+\hat{a}^\dagger\,\hat{\hat{a}}+\hat{a}\,\hat{a}^\dagger+\hat{a}\,\hat{a})\,(|0\rangle+|1\rangle)=\frac{1}{2}\,\frac{\hbar}{2\,m\omega}\,(\langle 0|+\langle 1|)\,(\hat{a}^\dagger\,\hat{a}^\dagger+2\,\hat{a}^\dagger\,\hat{a}+1+\hat{a}\,\hat{a})\,(|0\rangle+|1\rangle)=$

$\frac{\hbar}{4\,m\omega}\,((\langle 0|+\langle 1|)(\,2\,\hat{a}^\dagger\,\hat{a}+1)\,(|0\rangle+|1\rangle)=\frac{\hbar}{2\,m\omega}\,((\langle 0|+\langle 1|)(\hat{a}^\dagger\,\hat{a})\,(|0\rangle+|1\rangle)+\frac{1}{2}\,\frac{\hbar}{2\,m\omega}\,((\langle 0|+\langle 1|)(\,1)\,(|0\rangle+|1\rangle)=\frac{\hbar}{2\,m\omega}\,\langle 1|\hat{a}^\dagger\,\hat{a}|1\rangle$
$+\frac{\hbar}{2\,m\omega}=\frac{\hbar}{2\,m\omega}\,\langle 1|1\rangle+\frac{\hbar}{2\,m\omega}=\frac{\hbar}{m\omega}$

d) $\langle\hat{p}^2\rangle$ and check the Heisenberg uncertainty principle

$\hat{p}^2=-\frac{m\hbar\omega}{2}(\hat{a}^\dagger-\hat{a})(\hat{a}^\dagger-\hat{a})=-\frac{m\hbar\omega}{2}(\,\hat{a}^\dagger\hat{a}^\dagger-\hat{a}^\dagger\,\hat{a}-\hat{a}\,\hat{a}^\dagger+\hat{a}\,\hat{a}\,)=-\frac{m\hbar\omega}{2}(\,\hat{a}^\dagger\hat{a}^\dagger-\hat{a}^\dagger\,\hat{a}-1-\hat{a}^\dagger\,\hat{a}+\hat{a}\,\hat{a}\,)$

$=-\,\frac{m\hbar\omega}{2}(\hat{a}^\dagger\hat{a}^\dagger-2\,\hat{a}^\dagger\,\hat{a}-1+\hat{a}\,\hat{a}\,)$

$\langle\hat{p}^2\rangle=-\frac{1}{2}\,\frac{m\hbar\omega}{2}((\langle 0|+\langle 1|)(\,\hat{a}^\dagger\,\hat{a}^\dagger-\hat{a}^\dagger\,\hat{\hat{a}}-\hat{a}\,\hat{a}^\dagger+\hat{a}\,\hat{a})\,(|0\rangle+|1\rangle)=-\frac{1}{2}\,\frac{m\hbar\omega}{2}\,(\langle 0|+\langle 1|)\,(\hat{a}^\dagger\,\hat{a}^\dagger-2\,\hat{a}^\dagger\,\hat{a}-1+\hat{a}\,\hat{a})\,(|0\rangle+|1\rangle)=$

$\frac{m\hbar\omega}{4}\,((\langle 0|+\langle 1|)(\,2\,\hat{a}^\dagger\,\hat{a}+1)\,(|0\rangle+|1\rangle)=\frac{m\hbar\omega}{2}\,((\langle 0|+\langle 1|)(\hat{a}^\dagger\,\hat{a})\,(|0\rangle+|1\rangle)+\frac{1}{2}\,\frac{m\hbar\omega}{2}\,((\langle 0|+\langle 1|)(\,1)\,(|0\rangle+|1\rangle)=\frac{m\hbar\omega}{2}\,\langle 1|\hat{a}^\dagger\,\hat{a}|1\rangle$
$+\frac{m\hbar\omega}{2}=\frac{m\hbar\omega}{2}\,\langle 1|1\rangle+\frac{m\hbar\omega}{2}=m\hbar\omega$

$\mathbf{\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}}\qquad \mathbf{\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}}$

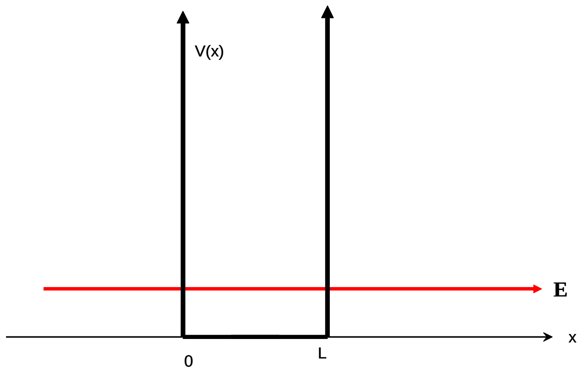
$\Delta x\Delta p=\sqrt{(\langle\hat{x}^2\rangle-\langle\hat{x}\rangle^2)\,(\langle\hat{p}^2\rangle-\langle\hat{p}\rangle^2)}=\sqrt{\left(\frac{\hbar}{m\omega}-\left(\sqrt{\frac{\hbar}{2\,m\omega}}\right)^2\right)(m\hbar\omega-0)}=\sqrt{\frac{\hbar}{2\,m\omega}\,m\hbar\omega}=\frac{\hbar}{2}\,\sqrt{2}>\frac{\hbar}{2}$ so it obeys heisenberg

e) $\langle\hat{H}\rangle$

We know H is diagonal in this representation so

$\langle\hat{H}\rangle = \frac{1}{2}(\langle 0|+\langle 1|)\,\hat{H}(|0\rangle+|1\rangle) = \frac{1}{2}(\langle 0|\hat{H}|0\rangle+\langle 1|\,\hat{H}|1\rangle) = \frac{1}{2}\,(\frac{1}{2}\,\hbar\omega + \frac{3}{2}\,\hbar\omega) = \hbar\omega$

3) a) You have a wave function for a particle in a box with sides at 0 and L as in the picture (careful, the edges are at 0 and L The wave function is in a mixture of the n=1 and n=2 energy eigenstates, i.e. $|\alpha,t=0\rangle = \frac{1}{\sqrt{2}}\,(|1\rangle + |2\rangle)$. A parameter you might want to use is $\hbar\omega = \frac{\hbar^2\pi^2}{2\,mL^2}$. Find the time dependence of $\langle\hat{x}\rangle$. Looking at the form of $\langle\hat{x}\rangle$ is it legitimate to think of the particle as rattling back and forth?



$|n\rangle = \sqrt{\frac{2}{L}}\,\sin\frac{n\pi x}{L}$ $\quad E_n = \frac{n^2\hbar^2\pi^2}{2\,mL^2}$ $\quad \hat{H}|n\rangle = \frac{n^2\hbar^2\pi^2}{2\,mL^2}|n\rangle$ For ease lets set $\hbar\omega = \frac{\hbar^2\pi^2}{2\,mL^2}$ $E_n = n^2\,\hbar\omega$. ω here is just a parameter and may have nothing to do with the oscillation frequency

$|\alpha,t\rangle = e^{\frac{i}{\hbar}\hat{H}\,t}\,\frac{1}{\sqrt{2}}\,(|1\rangle + |2\rangle) = \frac{1}{\sqrt{2}}\,(\,e^{i1^2\,\omega t}|1\rangle + e^{i2^2\,\omega t}|2\rangle) = \frac{1}{\sqrt{2}}\,(\,e^{i\omega t}|1\rangle + e^{4\,i\omega t}|2\rangle)$

$\langle\hat{x}\rangle = \frac{1}{2}\,(e^{-i\omega t}\,\langle 1| + e^{-4\,i\omega t}\,\langle 2|)x(\,e^{i\omega t}|1\rangle + e^{4\,i\omega t}|2\rangle) = \langle 1\,|\,\hat{x}\,|\,1\rangle + \langle 2\,|\,\hat{x}\,|\,2\rangle + e^{3\,i\omega t}\langle 1\,|\,\hat{x}\,|\,2\rangle + e^{-3\,i\omega t}\langle 2\,|\,\hat{x}\,|\,1\rangle$

now we will work in position representation

$\langle x|1\rangle = \sqrt{\frac{2}{L}}\,\sin\frac{\pi x}{L}$ $\quad \langle x|2\rangle = \sqrt{\frac{2}{L}}\,\sin\frac{2\pi x}{L}$ $\quad$ we will nee integrals like $\int_0^L dx\,x\,\sin\frac{n\pi x}{L}\,\sin\frac{m\pi x}{L}$

$\langle 1\,|\,\hat{x}\,|\,1\rangle = \frac{2}{L}\,\int_0^L x\sin(\frac{\pi x}{L})\sin(\frac{\pi x}{L})\,dx = 2\,L\int_0^1 y\sin(\pi\,y)\sin(\pi\,y)\,dy = \frac{2}{L}\,\frac{1}{4} = \frac{1}{2\,L}$

$\langle 2\,|\,\hat{x}\,|\,2\rangle = \frac{2}{L}\,\int_0^L x\sin(\frac{2\pi x}{L})\sin(\frac{2\pi x}{L})\,dx = 2\,L\int_0^1 y\sin(2\,\pi\,y)\sin(2\,\pi\,y)\,dy = \frac{2}{L}\,\frac{1}{4} = \frac{1}{2\,L}$

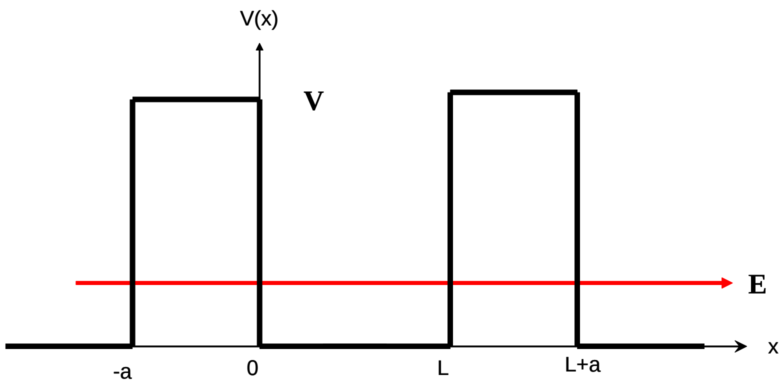
$\langle 1\,|\,\hat{x}\,|\,2\rangle = \frac{2}{L}\,\int_0^L x\sin(\frac{\pi x}{L})\sin(\frac{2\pi x}{L})\,dx = 2\,L\int_0^1 y\sin(\pi\,y)\sin(2\,\pi\,y)\,dy = \frac{2}{L}(-\frac{8}{9\pi^2}) = -\frac{16}{9\pi^2\,L} = \langle 2\,|\,\hat{x}\,|\,1\rangle$

$\langle\hat{x}\rangle = \frac{1}{2\,L} + \frac{1}{2\,L} - \frac{16}{9\,\pi^2\,L}(e^{3\,i\omega t} + e^{-3\,i\omega t}) = \frac{1}{L} - \cos 3\omega t\,\frac{32}{9\,\pi^2\,L} = \frac{1}{L}(1 - \frac{32}{9\,\pi^2}\cos 3\omega t)$

b) Now for the potential below calculate the transmission coefficient for the n=0 and n=1 states to penetrate through one of the barriers. Use for the energy, the average energy of the n=1 and n=2 states of from part a. That is use $E = \frac{1}{2}\left(\frac{1^2\hbar^2\pi^2}{2\,\mathrm{mL}^2}+\frac{2^2\hbar^2\pi^2}{2\,\mathrm{mL}^2}\right)= \frac{1}{2}\left(\hbar\omega + 4\,\hbar\omega\right) = \frac{5}{2}\,\hbar\omega$; L=10 fm (1 fm = 10^{-15} m); a=2 fm; mc^2=100 MeV; V=500 MeV

Remember $\hbar$c=200 MeV fm so we can calculate $\hbar\omega$.

$\hbar\omega = \frac{\hbar^2\pi^2}{2\,\mathrm{mL}^2} = \frac{\hbar^2 c^2\pi^2}{2\,\mathrm{mc}^2 L^2} = \frac{(200\,\mathrm{MeV\ fm})^2\pi^2}{2\cdot 100\,\mathrm{MeV}\ (10\,\mathrm{fm})^2}$ ~20 MeV



$$\frac{1}{T} = 1 + \frac{V^2}{4E(V-E)}\sinh^2(2\,\kappa_2\,a) \quad \kappa_2 = \sqrt{\frac{2m}{\hbar^2}(V-E)}$$

$$\frac{V^2}{4E(V-E)} = \frac{500^2\ \mathrm{MeV}^2}{4\,\frac{5}{2}\,20\,\mathrm{MeV}\left(500\,\mathrm{MeV}-\frac{5}{2}\,20\,\mathrm{MeV}\right)} \sim 2.8$$

$$\kappa_2 = \sqrt{\frac{2m}{\hbar^2}(V-E)} = \kappa_2 = \sqrt{\frac{2\,\mathrm{mc}^2}{(\hbar c)^2}(V-E)} = \sqrt{\frac{2\cdot 100\,\mathrm{MeV}}{(200)^2\ \mathrm{MeV}^2\ \mathrm{fm}^2}\left(500\,\mathrm{MeV}-\frac{5}{2}\,20\,\mathrm{MeV}\right)} = \frac{1.5}{\mathrm{fm}}$$

$$\sinh^2\left(2\cdot\frac{1.5}{\mathrm{fm}}\,2\,\mathrm{fm}\right) = \sinh^2(6)=40700$$

$$\frac{1}{T}=1+2.8\cdot 40700=113961 \qquad T = 9\,\mathrm{x}10^{-6}$$

c) Now lets assume you can figure out the frequency that the particle hits the wall assuming that the wave functions do not change much from part a). Figure out the lifetime of the state.

$$P=fT=2\frac{\omega}{2\pi}T$$

$$\tau = \frac{1}{P} = \frac{\pi}{T\omega} \quad \text{so} \quad \tau c = \frac{\pi\hbar c}{T\hbar\omega} = \frac{\pi(200\,\mathrm{MeV\cdot fm})}{9\,\mathrm{x}10^{-6}\ 20\,\mathrm{MeV}} = 3.5\,\mathrm{x}10^6\ \mathrm{fm} \qquad \tau = \frac{3.5\,\mathrm{x}10^6\ \mathrm{fm}}{3\,\mathrm{x}10^8\ m/s}\ \frac{1\ m}{10^{15}\ \mathrm{fm}} = 1.17\,\mathrm{x}10^{-17}\mathrm{s}$$

maybe we should use 3ω for the frequency instead of ω, in which case it would be 0.4x10^{-17}s

Here are some constants :

$$1\ \text{fm} = 10^{-15}\ m$$

$\hbar c = 200\ \text{MeV} \cdot \text{fm}$

$\hbar = 1.05457148 \times 10^{-34}\ J \cdot s = 6.5821\ \times 10^{-16}\ \text{eV} \cdot s$

$c = 3 \times 10^{8}\ m / s$

$1\ \text{eV} = 1.6 \times 10^{-19}\ J$

Here are 4 integrals you may need. For 2 − 4 *I* calculate the definite integral between 0 and 1 for you as well as the indefinite integral

1) $\int x \sin(n \pi x) \sin(m \pi x)\, dx$

Integrate[x * Sin[n * Pi * x] * Sin[m * Pi * x], x]

$$\frac{\frac{\cos((m-n)\pi x)+(m-n)\pi x \sin((m-n)\pi x)}{(m-n)^2} - \frac{\cos((m+n)\pi x)+(m+n)\pi x \sin((m+n)\pi x)}{(m+n)^2}}{2\pi^2}$$

2) $\int x \sin(\pi x) \sin(\pi x)\, dx$

Integrate[x * Sin[Pi * x] * Sin[Pi * x], x]

$$-\frac{\cos(2\pi x) + 2\pi x (\sin(2\pi x) - \pi x)}{8\pi^2}$$

Integrate[x * Sin[Pi * x] * Sin[Pi * x], {x,0,1}]

$$\frac{1}{4}$$

3) $\int x \sin(\pi x) \sin(2\pi x)\, dx$

Integrate[x * Sin[Pi * x] * Sin[2 * Pi * x], x]

$$\frac{12\pi x \sin^3(\pi x) + 9\cos(\pi x) - \cos(3\pi x)}{18\pi^2}$$

Integrate[x * Sin[Pi * x] * Sin[2 * Pi * x], {x,0,1}]

$$-\frac{8}{9\pi^2}$$

4) $\int x \sin(2\pi x) \sin(2\pi x)\, dx$

Integrate[x * Sin[2 * Pi * x] * Sin[2 * Pi * x], x]

$$-\frac{\cos(4\pi x) + 4\pi x(\sin(4\pi x) - 2\pi x)}{32\pi^2}$$

Integrate[x * Sin[2 * Pi * x] * Sin[2 * Pi * x], {x, 0, 1}]

$$\frac{1}{4}$$