matrix manipulations

matrix elements (which will be operators) are labeled $a_{\text{row, column}}$ as follows

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

kets will be column vectors $\begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$ we will often just label this as $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

bra's will be row vectors $\begin{pmatrix} c_{11} & c_{12} & c_{13} \end{pmatrix}$ or $\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$

many operations, like adding (subtracting), multiplying by a constant, and taking the real or imaginary part means just doing it term by term. You can add two matrices, two column vectors, but you cannot add a matrix and a column vector or add a row vector to a column vector – just like you cannot add a bra to a ket or add a operator to a ket

matrix multiplication

- we will always multiply rows by columns and put the answer in the row/column used. First lets multiply a row by a column which is like multiplying a bra by a ket. $\langle a | b \rangle$ which makes a number

$$\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = c_1 b_1 + c_2 b_2 + c_3 b_3$$

- Multiplying column vector by a matrix is like operating on a ket

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_{11} b_1 + a_{12} b_2 + a_{13} b_3 \\ a_{21} b_1 + a_{22} b_2 + a_{23} b_3 \\ a_{31} b_1 + a_{32} b_2 + a_{33} b_3 \end{pmatrix}$$

- Multiplying two matrices is like combining two operators Here I have only colored in a a few examples

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}d_{11}+a_{12}d_{21}+a_{13}d_{31} & a_{11}d_{12}+a_{12}d_{22}+a_{13}d_{32} & a_{11}d_{13}+a_{12}d_{23}+a_{13}d_{33} \\ a_{21}d_{12}+a_{22}d_{22}+a_{23}d_{32} & a_{21}d_{12}+a_{22}d_{22}+a_{23}d_{32} & a_{21}d_{13}+a_{22}d_{23}+a_{23}d_{33} \\ a_{31}d_{13}+a_{32}d_{23}+a_{33}d_{33} & a_{31}d_{12}+a_{32}d_{22}+a_{33}d_{32} & a_{31}d_{13}+a_{32}d_{23}+a_{33}d_{33} \end{pmatrix}$$

- Multiplying a row by a matrix is like combining a bra and a matrix. We will ususally not do this

$$\begin{pmatrix} c_1 & c_1 & c_1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$\begin{pmatrix} c_1a_{11}+c_2a_{21}+c_3a_{31} & c_1a_{12}+c_2a_{22}+c_3a_{32} & c_1a_{13}+c_2a_{23}+c_3a_{33} \end{pmatrix}$$

Hermitian conjugate

$$\text{if } \hat{A} \doteq \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then

$$\hat{A}^{\dagger} \doteq \begin{pmatrix} a_{11}^* & a_{21}^* & a_{31}^* \\ a_{12}^* & a_{22}^* & a_{32}^* \\ a_{13}^* & a_{23}^* & a_{33}^* \end{pmatrix}$$

So we just interchange the rows and columns and then take the complex conjugate of everything

Determinant of a matrix

- The determinant of a matrix **A** denoted as det **A** is explained at
  - http://mathworld.wolfram.com/Determinant.html
  - http://mathforum.org/library/drmath/view/51440.html
- Here are a couple tricks to figure out the determinant
- For a 2x2 matrix its easy

$$\det(\widehat{\mathrm{A}}) \doteq \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Note that this is just one diagonal (red) minus the other diagonal (blue)
Note also that the matrix with a | | around it means the determinant

Determinant 3x3 of a matrix

- For a 3x3 matrix we multiply the diagonal numbers together. Notice that when you come to the end of a row/column you just wrap around

$$\overset{\frown}{A} \doteq \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

So we start with $a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} +$

Determinant of 3x3 a matrix

Then we go the other way but put a negative sign

$$\widehat{A} \doteq \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$-(a_{13}\, a_{22}\, a_{31} + a_{11}\, a_{23}\, a_{32} + a_{12}\, a_{21}\, a_{33})$

So finally we have det A=

$a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - (a_{13}\, a_{22}\, a_{31} + a_{11}\, a_2 3\, a_{32} + a_{12}\, a_{21}\, a_{33})$

Finding the eigenvalues and eigenvectors of a 2x2 matrix

When we want to find the eigenvalues and eigenkets of an operator in matrix form we can just find the eigenvalues and eigenvectors as follows

Let's start with a matrix **A** and a column vector **X**. If **X** is an eigenvector of **A** then we can write

$\mathbf{AX}=\lambda\mathbf{X}$ where λ is a number (it will of course turn out to be several numbers - the eigenvalues.) **I** will stand for the identity matrix - i.e. it has 1's down the diagonal, then $(\mathbf{A}-\lambda\,\mathbf{I})\mathbf{X}=0$

Now there is a nice way to solve such things. We can just solve det($(\mathbf{A}-\lambda\,\mathbf{I})\mathbf{X}$)=0 for λ. Lets first assume that **A** is a 2x2 matrix then

$$\det\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0 \qquad \text{OR} \qquad \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}=0$$

$$a_{11}a_{22}-\lambda a_{11}-\lambda a_{22}+\lambda^2-a_{12}a_{21}=0$$

$$\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})=0$$

$$\lambda=\frac{(a_{11}+a_{22})\pm\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}}{2}=\frac{1}{2}\left[(a_{11}+a_{22})\pm\sqrt{4a_{12}a_{21}+(a_{11}-a_{22})^2}\right]$$

Finding the eigenvectors

Now we know the two eigenvalues $\lambda_{\pm} = \frac{1}{2}\left[(a_{11}+a_{22}) \pm \sqrt{4a_{12}a_{21}+(a_{11}-a_{22})^2}\right]$

we have to solve $(\mathbf{A}-\lambda\ \mathbf{I})\mathbf{X} = \mathbf{0}$ for $\mathbf{X}$ where $\lambda = \lambda_+$ and λ_-. Lets let $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} (a_{11}-\lambda)x_1 + a_{12}x_2 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(a_{11}-\lambda)x_1 + a_{21}x_2 = 0 \qquad a_{21}x_1 + (a_{22}-\lambda)x_2 = 0$ it turns out these are not independent

So all we learn is that $x_2 = -\frac{(a_{11}-\lambda)}{a_{21}}x_1$

but we have the normalization condition that $x_1^2 + x_2^2 = 1$ so let $x_1 = c$ then we have

$$c^2\left(1+\left(\frac{a_{11}-\lambda}{a_{21}}\right)^2\right) = 1 \qquad \text{so } x_1 = \text{c} = \frac{1}{\sqrt{1+\left(\frac{a_{11}-\lambda}{a_{21}}\right)^2}} = \frac{a_{21}}{\sqrt{a_{21}^{\ 2}+\left(a_{11}-\lambda\right)^2}}$$

and $x_2 = -\frac{(a_{11}-\lambda)}{a_{21}\sqrt{1+\left(\frac{a_{11}-\lambda}{a_{21}}\right)^2}} = -\frac{(a_{11}-\lambda)}{\sqrt{a_{21}^{\ 2}+\left(a_{11}-\lambda\right)^2}}$

Unfortunately this does not simplify much and we have two sets of eigenvalue+eigenvector

$$\lambda_{\pm} = \frac{1}{2}\left[(a_{11} + a_{22}) \pm \sqrt{4a_{12}a_{21} + (a_{11} - a_{22})^2} \right]$$

$$\mathbf{X}_{\pm} = \begin{pmatrix} \dfrac{a_{21}}{\sqrt{a_{21}{}^2 + \left(a_{11} - \lambda_{\pm}\right)^2}} \\ \dfrac{-(a_{11} - \lambda_{\pm})^2}{\sqrt{a_{21}{}^2 + \left(a_{11} - \lambda_{\pm}\right)^2}} \end{pmatrix}$$

Diagonalizing a matrix

We will talk about changing basis later, but let me just tell you that you can change the basis on an operator to diagnolize it. This is simple. The eigenvalues are the same

$$\lambda_{\pm} = \frac{1}{2}\left[(a_{11} + a_{22}) \pm \sqrt{4a_{12}a_{21} + (a_{11} - a_{22})^2)} \right]$$

The matrix becomes just

$$\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

and the eigenvectors are

$$\mathbf{X}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and } \mathbf{X}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$