Notes for Quantum Mechanics

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Lecture 19

The simple harmonic oscillator

So far our hamiltonians have been pretty simple. We have had the free particle hamiltonian with a kinetic energy term. and a spin hamiltonian with a spin-magnetic field interaction. We will now take up a case where we have both the kinetic energy term and a x-dependent potential. A classical simple harmonic oscillator (SHO) has a restoring for F=-kx which is proportional to the distance, giving us a potential energy term $V(x) = \frac{1}{2}\,\mathrm{kx}^2$. It is the potential given by a stretched spring. Where is such a potential useful in the quantum world. It turns out, just about everywhere. The strong interaction makes quarks stick together in the form of protons and neutrons (hadrons) - a good approximation to the potential is just a simple harmonic oscillator potential. Atoms which jiggle around in a crystal lattice can be thought of as held in place by a bunch of springs. In fact if one expands any potential in a power series, the first symmetric term is just the SHO.

It turns out that the SHO is a problem which is readily solvable (unlike just about all other potentials - the coulomb potential being another one which is semi-solvable as you will see) so we tend to model anything force which is attractive with a SHO or coulomb potential. OK so lets pretend we have two quarks and they are held together by the strong interaction - i.e. with a force which grows linearly with distance. (Heavy quarks like bottom are well modeled by a 3-D version of the model we are now going to build ; we will stay in 1-D to make things easy. BTW one reason you don't see free quarks is that the energy you put into two quarks to pull them apart is large - large enough that is easier to pull a pair of quarks out of the vacuum, thereby giving you two pairs). These quarks can also have kinetic energy so we will start out our model with a hamiltonian as follows:

$H = \frac{p^2}{2m}$+V(x) where V(x)=$\frac{1}{2}\,\mathrm{m}\omega^2\,x^2$ m is the mass of the quark, and $k = \mathrm{m}\omega^2$ is the "spring" constant which will have to be measured in an experiment. Note that $\frac{1}{2}\,\mathrm{m}\omega^2\,x^2$ has the units of energy if m is mass and ω is a frequency. Now we will just assume now that the p and x will be operators so

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}\,\mathrm{m}\omega^2\,\hat{x}^2$ in the quantum world. Now its a pain working with two quarks, so we can always work in a the CM system where the m is a reduced mass and just pretend that its just one quark moving around a fixed point. If you dont know what I am talking about, go back to some freshman physics book and look up reduced mass.

OK. Now we have a hamiltonian and let our quark be represented by some state $|\alpha\rangle$. Later we will have to specify some sort of initial condition and then find the time dependence.

The brute force way is to write this thing in the x representation as

$\frac{-\hbar^2}{2m}\frac{\mathrm{d}\psi(x)}{\mathrm{dx}^2}+\frac{1}{2}\,\mathrm{m}\omega^2\,\hat{x}^2\psi(\mathrm{x})=\mathrm{E}\psi(\mathrm{x})$ where $\psi(\mathrm{x})=\langle \mathrm{x}|\alpha\rangle$. It works, its a pain. There is a much cleverer and easier way of doing it using raising and lowering operators.

Lets define a couple of new operators as follows:

$\hat{a}=\sqrt{\frac{\mathrm{m}\omega}{2\hbar}}\left(\hat{x}+\frac{i\,\hat{p}}{\mathrm{m}\omega}\right)$ and $\hat{a}^\dagger=\sqrt{\frac{\mathrm{m}\omega}{2\hbar}}\left(\hat{x}-\frac{i\,\hat{p}}{\mathrm{m}\omega}\right)$ [we will call $\hat{a}$ a lowering operator and $\hat{a}^\dagger$ a raising operator)

The reverse eqns for reference are $\hat{x}=\sqrt{\frac{\hbar}{2\,m\omega}}\,(\hat{a}^\dagger+\hat{a})$ and $\hat{p}=i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^\dagger-\hat{a})$

Then remembering that $[\hat{x},\hat{p}]=i\hbar$ $\quad [\hat{a},\hat{a}^\dagger]=\frac{i}{2\hbar}\,(-[\hat{x},\hat{p}]+[\hat{p},\hat{x}])=1$

We can also define the number operator $\hat{N}=\hat{a}^\dagger\,\hat{a}=\frac{m\omega}{2\hbar}\left(\hat{x}^2+\frac{\hat{p}^2}{m^2\,\omega^2}+\frac{i}{m\omega}[\hat{x},\,\hat{p}]\right)=\frac{m\omega}{2\hbar}\,\hat{x}^2+\frac{\hat{p}^2}{2\,\hbar m\omega}-\frac{1}{2}=\frac{\hat{H}}{\hbar\omega}-\frac{1}{2}$

So $\hat{H}=\hbar\omega\left(\hat{N}+\frac{1}{2}\right)$

We see immediately that $[\hat{H},\hat{N}]=0$ so we can have eigenkets of $\hat{H}$ which are also eigenkets of $\hat{N}$ and we will label the eigenkets of both $\hat{H}$ and $\hat{N}$ as $|n\rangle$ so that

$\hat{N}|n\rangle=n|n\rangle$ and it follows that $\hat{H}\,|n\rangle=\hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$ so the eigenvalues of $\hat{H}$ are $E_n=\hbar\omega\left(n+\frac{1}{2}\right)$

Now why do we call these thing raising and lowering operators. Lets see:

$[\hat{N},\hat{a}]=[\hat{a}^\dagger\,\hat{a},\hat{a}]=\hat{a}^\dagger[\hat{a},\hat{a}]+[\hat{a}^\dagger,\hat{a}]\hat{a}=-\hat{a}$ and $[\hat{N},\hat{a}^\dagger]=[\hat{a}^\dagger\,\hat{a},\hat{a}^\dagger]=\hat{a}^\dagger[\hat{a},\hat{a}^\dagger]+[\hat{a}^\dagger,\hat{a}^\dagger]\hat{a}=\hat{a}^\dagger$

So $\hat{N}\hat{a}^\dagger|n\rangle=(\hat{a}^\dagger+\hat{a}^\dagger\hat{N})|n\rangle=\hat{a}^\dagger(n+1)|n\rangle=(n+1)\hat{a}^\dagger|n\rangle$ so $\hat{a}^\dagger|n\rangle\sim|n+1\rangle$ i.e. it raises it a step! There is a ~ there because there is still a normalization constant to figure out. Similarly

$\hat{N}\hat{a}|n\rangle=(-\hat{a}+\hat{a}\hat{N})|n\rangle=\hat{a}(n-1)|n\rangle=(n-1)\hat{a}|n\rangle$ so $\hat{a}|n\rangle\sim|n-1\rangle$ i.e. its a lowering operator.

Sometimes these things are call annihilation and creation operators because they either create of destroy one quantum of energy. Now lets figure out the normalization constant.

let c be the constant so $\hat{a}|n\rangle=c|n-1\rangle$. We want both $\langle n|n\rangle=1$ and $\langle n-1|n-1\rangle=1$ so

$\langle n|\,\hat{a}^\dagger\,\hat{a}\,|n\rangle=|c|^2\,\langle n-1|n-1\rangle=|c|^2$ but $\langle n|\,\hat{a}^\dagger\,\hat{a}\,|n\rangle=\langle n|\,\hat{N}\,|n\rangle=n\langle n|n\rangle=n$ so $c=\sqrt{n}$

Also we know $[\hat{a},\hat{a}^\dagger]=1 \;\rightarrow\; \hat{a}\,\hat{a}^\dagger=1+\hat{a}^\dagger\,\hat{a}=1+\hat{N}$

let $\hat{a}^\dagger|n\rangle=c|n+1\rangle$ so $\langle n|\,\hat{a}\,\hat{a}^\dagger\,|n\rangle=|c|^2\,\langle n+1|n+1\rangle=|c|^2$ but $\langle n|\,\hat{a}\,\hat{a}^\dagger\,|n\rangle=\langle n|\,\hat{N}+1\,|n\rangle=(n+1)\langle n|n\rangle=n+1$ so $c=\sqrt{n+1}$ and finally summing up

$\hat{a}^\dagger|n\rangle=\sqrt{n+1}\,|n+1\rangle$ and $\hat{a}|n\rangle=\sqrt{n}\,|n-1\rangle$

Now if we can just figure out some lowest energy state, we can just bootstrap our way up by using the raising operators!

So lets see if we can formulate an argument.

First we can show that n must be an integer: For the moment lets call $\hat{a}|n\rangle=|\alpha\rangle$. Now we dont know if this thing is normalized (it isn't as we know from above) but we do know that $\langle\alpha|\alpha\rangle\geq 0$ $\rightarrow$ $n=\langle n\,|\,\hat{N}\,|\,n\rangle=\langle n|\hat{a}^\dagger\,\hat{a}|n\rangle=\langle\alpha|\alpha\rangle\geq 0$ so $n\geq 0$

Next we know that if we start is some $|n\rangle$ then we can use the lowering operator as follows

$\hat{a}|n\rangle=\sqrt{n}\,|n-1\rangle$

$\hat{a}^2|\text{n}\rangle=\sqrt{n(n-1)}\;|\text{n-2}\rangle$

$\hat{a}^3|\text{n}\rangle=\sqrt{n(n-1)\,(n-2)}\;|\text{n-3}\rangle$ etc. This can keep going forever, UNLESS at some point the number inside the ket is zero, then the series will terminate.

Putting this together with the fact that n≥0 means that the series MUST terminate at n=0. Therefore n=0 is the lowest energy state where we want to start our bootstrap.

So the ground state is $|0\rangle$ with energy eigenvalue $E_0 = \frac{1}{2}\hbar\omega$ It is striking that the lowest energy state is NOT zero! This

has implications as we ask how what things look like at zero temperature - what the state of the vacuum is, dark energy, the casmir force etc etc.

Now we can just work our way up to all the other eigenstataes:

$|1\rangle=\hat{a}^\dagger\,|0\rangle \quad E_1 = \frac{3}{2}\hbar\omega$

$|2\rangle=\frac{\hat{a}^\dagger}{\sqrt{2}}\,|1\rangle = \frac{(\hat{a}^\dagger)^2}{\sqrt{2!}}\,|0\rangle \quad E_2 = \frac{5}{2}\hbar\omega$

$|3\rangle=\frac{\hat{a}^\dagger}{\sqrt{3}}\,|2\rangle = \frac{(\hat{a}^\dagger)^3}{\sqrt{3!}}\,|0\rangle \quad E_2 = \frac{7}{2}\hbar\omega$

$|\text{n}\rangle= \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle \quad E_n = \left(n+\frac{1}{2}\right)\hbar\omega$

Matrix representation of SHO operators

Now lets look at some particular ways of writing some of these things down. Lets first look at the matrix representation of the SHO operators, using the eigenkets $|\text{n}\rangle$ as the basis. The lowest number n can be is zero, so the rows and columns will be numbered, 0,1,2,... A space with a □ is a 0.

$$\hat{N}\doteq\left\langle n' \mid \hat{N} \mid n''\right\rangle\doteq\begin{pmatrix} 0 & □ & □ & □ & □ \\ □ & 1 & □ & □ & □ \\ □ & □ & 2 & □ & □ \\ □ & □ & □ & □ & □ \\ □ & □ & □ & □ & n' \end{pmatrix} \qquad \hat{H}\doteq\left\langle n' \mid \hat{H} \mid n''\right\rangle\doteq\begin{pmatrix} 1/2 & □ & □ & □ & □ \\ □ & 3/2 & □ & □ & □ \\ □ & □ & 5/2 & □ & □ \\ □ & □ & □ & □ & □ \\ □ & □ & □ & □ & n'+1/2 \end{pmatrix}\hbar\omega$$

$$\left\langle n' \mid \hat{a} \mid n\right\rangle\doteq\sqrt{n}\;\delta_{n',n-1}\doteq\begin{pmatrix} □ & \sqrt{1} & □ & □ & □ \\ □ & □ & \sqrt{2} & □ & □ \\ □ & □ & □ & \sqrt{3} & □ \\ □ & □ & □ & □ & □ \\ □ & □ & □ & □ & □ \end{pmatrix} \qquad \left\langle n' \mid \hat{a}^\dagger \mid n\right\rangle\doteq\sqrt{n+1}\;\delta_{n',n+1}\doteq\begin{pmatrix} □ & □ & □ & □ & □ \\ \sqrt{1} & □ & □ & □ & □ \\ □ & \sqrt{2} & □ & □ & □ \\ □ & □ & \sqrt{3} & □ & □ \\ □ & □ & □ & □ & □ \end{pmatrix}$$

and using $\hat{x}=\sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}}\,\left(\hat{a}^\dagger+\hat{a}\right)$ and $\hat{p}=\mathrm{i}\sqrt{\frac{\mathrm{m}\hbar\omega}{2}}\,\left(\hat{a}^\dagger-\hat{a}\right)$ we get

$$\langle n' \mid \hat{x} \mid n\rangle \doteq \sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}}\,(\sqrt{n+1}\;\delta_{n',n+1}+\sqrt{n}\;\delta_{n',n-1})=\sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}}\begin{pmatrix} \square & \sqrt{1} & \square & \square & \square \\ \sqrt{1} & \square & \sqrt{2} & \square & \square \\ \square & \sqrt{2} & \square & \sqrt{3} & \square \\ \square & \square & \sqrt{3} & \square & \square \\ \square & \square & \square & \square & \square \end{pmatrix}$$

$$\langle n' \mid \hat{p} \mid n\rangle \doteq i\sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}}\,(\sqrt{n+1}\;\delta_{n',n+1}-\sqrt{n}\;\delta_{n',n-1})=\sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}}\begin{pmatrix} \square & -\sqrt{1} & \square & \square & \square \\ \sqrt{1} & \square & -\sqrt{2} & \square & \square \\ \square & \sqrt{2} & \square & -\sqrt{3} & \square \\ \square & \square & \sqrt{3} & \square & \square \\ \square & \square & \square & \square & \square \end{pmatrix}$$

Note that $\hat{x}$, $\hat{p}$, $\hat{a}$, $\hat{a}^\dagger$ all are not diagonal. This makes sense because they all do not commute with $\hat{N}$ (or $\hat{H}$). As a reminder, these matrices are all infinite dimensional. Lets look at how this works, I will only make my matrices 4x4, but in your mind let the rows and columns go to infinity. Lets take a look at the raising and lower operator and see what they do. The $|0\rangle$ has a 1 in the 0th row. $\hat{a}^\dagger\,|0\rangle \doteq \begin{pmatrix} \square & \square & \square & \square & \square \\ \sqrt{1} & \square & \square & \square & \square \\ \square & \sqrt{2} & \square & \square & \square \\ \square & \square & \sqrt{3} & \square & \square \\ \square & \square & \square & \square & \square \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \sqrt{1}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \doteq \sqrt{1}\,|1\rangle$As expected the raising operator moves the "1" up to the next row. Doing it again would give $\sqrt{2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \doteq \sqrt{2}\,|2\rangle$. Using the lowering operator would "lower" the row by one. The expectation value for energy for $|2\rangle$ is

$$\langle 2 \mid \hat{H} \mid 2\rangle=\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1/2 & \square & \square & \square \\ \square & 3/2 & \square & \square \\ \square & \square & 5/2 & \square \\ \square & \square & \square & 7/2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\hbar\omega = 5/2\,\hbar\omega = \left(2+\tfrac{1}{2}\right)\hbar\omega$$

We avoided solving the differential eqn which would give us the wave function in position representation - because it is a pain. It turns out, this raising and lowering operator also makes finding the wave functions easier. Remember we decided that n=0 was the ground state and we know $\hat{a}\,|0\rangle=\sqrt{0}\,|{-1}\rangle=0$. Writing this in position representation gives

$$\langle x' \mid \hat{a} \mid 0\rangle=\sqrt{\frac{\mathrm{m}\omega}{2\,\hbar}}\left\langle x' \,\middle|\, \hat{x}+\frac{i\,\hat{p}}{\mathrm{m}\omega} \,\middle|\, 0\right\rangle=\sqrt{\frac{\mathrm{m}\omega}{2\,\hbar}}\,\left(\langle x' \mid \hat{x} \mid 0\rangle+\frac{i}{\mathrm{m}\omega}\,\langle x' \mid \hat{p} \mid 0\rangle\right)$$

Now $\langle x' \mid \hat{x} \mid 0\rangle=\langle x' \mid \hat{x} \mid 0\rangle^{\dagger\dagger}=\langle 0 \mid \hat{x}^\dagger \mid x'\rangle^{\dagger}=\langle 0 \mid \hat{x} \mid x'\rangle^{\dagger}=\mathrm{x}'\langle 0 \mid x'\rangle^{\dagger}=\mathrm{x}'\langle \mathrm{x}'|0\rangle$ and $\langle x' \mid \hat{p} \mid 0\rangle=\frac{\hbar}{i}\,\frac{d}{\mathrm{dx}'}\langle x' \mid 0\rangle$ so

$\langle x' \mid \hat{a} \mid 0\rangle=\sqrt{\frac{\mathrm{m}\omega}{2\,\hbar}}\,(\mathrm{x}'\langle \mathrm{x}'|0\rangle+\frac{\hbar}{\mathrm{m}\omega}\,\frac{d}{\mathrm{dx}'}\langle \mathrm{x}'|0\rangle)$ and finally $\left(x'+x_0^2\,\frac{d}{\mathrm{dx}'}\right)\langle \mathrm{x}'|0\rangle=0$ where $x_0=\sqrt{\frac{\hbar}{\mathrm{m}\omega}}$

The solution to this is a gaussian:

$\langle x' \mid 0\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right)\exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right]$ (check it). We can just use the raising operator to find the rest:

$\langle x'|1\rangle = \langle x' \mid \hat{a}^\dagger \mid 0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left\langle x' \,\middle|\, \hat{x} - \frac{i\,\hat{p}}{m\omega} \,\middle|\, 0\right\rangle = \sqrt{\frac{m\omega}{2\hbar}}\,(\langle x' \mid \hat{x} \mid 0\rangle - \frac{i}{m\omega}\langle x' \mid \hat{p} \mid 0\rangle) = \sqrt{\frac{m\omega}{2\hbar}}\,(x'\langle x'|0\rangle - \frac{\hbar}{m\omega}\frac{d}{dx'}\langle x'|0\rangle)$

$= \left(\frac{1}{\sqrt{2}\,x_0}\right)\left(x' - x_0^2\,\frac{d}{dx'}\right)\langle x'|0\rangle$

We can going and in general $\langle x' \mid n\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{2^n\,n!}}\right)\left(\frac{1}{x_0^{n+1/2}}\right)\left(x' - x_0^2\,\frac{d}{dx'}\right)^n\exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right]$

$\langle x' \mid n\rangle = A_n\,\mathcal{H}_n(\xi)\,e^{-\xi^2}$ where $\xi = \frac{m\omega_0}{\hbar}x^2$, $A_n = \left(2^n\,n!\,\sqrt{\pi}\right)^{-1/2}$ and $\mathcal{H}_n(\xi)$ are the Hermite polynomials

so $\langle x'|0\rangle = A_0 e^{-\frac{\xi^2}{2}}$ $\langle x'|1\rangle = A_1\,2\,\xi e^{-\frac{\xi^2}{2}}$ $\langle x'|2\rangle = A_2\left(4\,\xi^2 - 2\right)e^{-\frac{\xi^2}{2}}$ etc

Lets figure out some stuff for the ground state $|0\rangle$

$\langle\hat{x}\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle\hat{a}^\dagger + \hat{a}\rangle = 0$ and $\langle\hat{p}\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,\langle\hat{a}^\dagger - \hat{a}\rangle = 0$

$\langle\hat{x}^2\rangle = \frac{\hbar}{2m\omega}\langle\hat{a}^\dagger\,\hat{a}^\dagger + \hat{a}^\dagger\,\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\,\hat{a}\rangle = \frac{\hbar}{2m\omega}\left\langle\hat{a}\,\hat{a}^\dagger\right\rangle = \frac{\hbar}{2m\omega}\langle 0|\,\hat{a}\,\hat{a}^\dagger\,|0\rangle = \frac{\hbar}{2m\omega}(0+1)\left\langle 1|\,1\right\rangle = \frac{\hbar}{2m\omega} = \frac{x_0^2}{2}$

$\langle\hat{p}^2\rangle = -\frac{m\hbar\omega}{2}\langle\hat{a}^\dagger\,\hat{a}^\dagger - \hat{a}^\dagger\,\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\,\hat{a}\rangle = \frac{m\hbar\omega}{2}\left\langle\hat{a}\,\hat{a}^\dagger\right\rangle = \frac{m\hbar\omega}{2}\langle 0|\,\hat{a}\,\hat{a}^\dagger\,|0\rangle = \frac{m\hbar\omega}{2}(0+1)\left\langle 1|\,1\right\rangle = \frac{m\hbar\omega}{2}$

The kinetic energy $\left\langle\frac{\hat{p}^2}{2m}\right\rangle = \frac{\hbar\omega}{4} = \frac{\langle\hat{H}\rangle}{2}$ and $\left\langle\frac{m\omega^2\,\hat{x}^2}{2}\right\rangle = \frac{m\omega^2}{2}\frac{\hbar}{2m\omega} = \frac{\hbar\omega}{4} = \frac{\langle\hat{H}\rangle}{2}$ so the energy is split for the ground state between the potential energy and kinetic energy as expected from the virial theorem

$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$ $\Delta x = \sqrt{\frac{m\hbar\omega}{2}}$ $\Delta x\,\Delta p = \frac{\hbar}{2}$ (i.e. it is at the minimum uncertainty by Heisenberg)

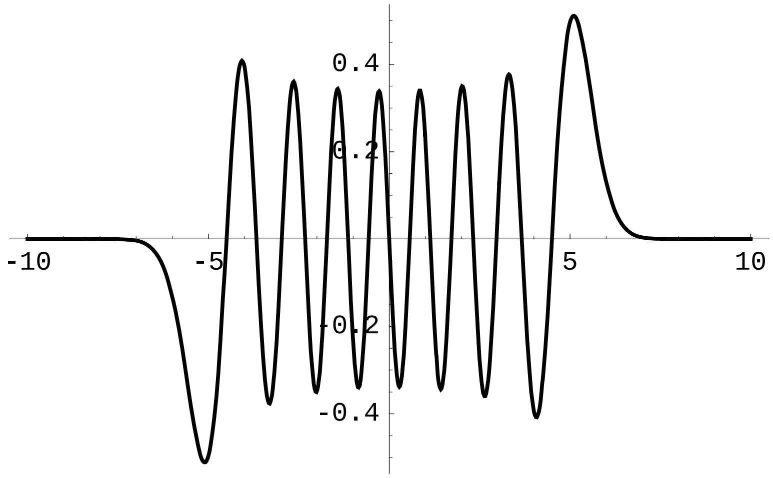
What to these wave functions look like?.

Here are a few

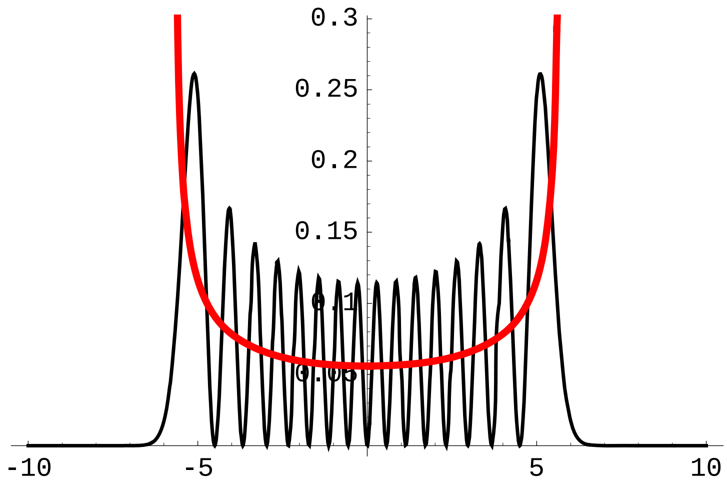
$\langle x'|0\rangle = \frac{e^{-\frac{x^2}{2}}}{\sqrt[4]{\pi}}$ $\langle x'|1\rangle = \frac{\sqrt{2}\,e^{-\frac{x^2}{2}}\,x}{\sqrt[4]{\pi}}$ $\langle x'|2\rangle = \frac{e^{-\frac{x^2}{2}}\,(2x^2-1)}{\sqrt{2}\,\sqrt[4]{\pi}}$ $\langle x'|3\rangle = \frac{e^{-\frac{x^2}{2}}\,x\,(2x^2-3)}{\sqrt{3}\,\sqrt[4]{\pi}}$ $\langle x'|4\rangle = \frac{e^{-\frac{x^2}{2}}\,(4(x^2-3)x^2+3)}{2\sqrt{6}\,\sqrt[4]{\pi}}$

I have plotted $\langle x'|n=15\rangle$. (See qm19work for the mathematica to figure out the wave functions and plot it)

$$\langle x'|15\rangle = \frac{e^{-\frac{x^2}{2}}\,x\,(2x^2(2(2x^2(2(8x^6-420x^4+8190x^2-75\,075)x^2+675\,675)-2\,837\,835)x^2+4\,729\,725)-2\,027\,025)}{30\,240\,\sqrt{715}\,\sqrt[4]{\pi}}$$



I have also plotted the probability (in black) and the classical probability (in red). Now note that the classical probability is really the mass that goes back and forth with time. The QM probability is the probability at t=0! So far there is no time dependence.



Let's now add the time dependence for the eigenstates.

$|n,\, t\rangle = \exp\left(-\frac{i\,\hat{H}\,t}{\hbar}\right)\left|n,\, t=0\right\rangle = \exp\left(-\frac{i\,(\hat{N}+1/2)\,\hbar\omega\,t}{\hbar}\right)\left|n,\, t=0\right\rangle = e^{-i\left(n+\frac{1}{2}\right)\omega\,t}|\text{n,t=0}\rangle$

From here we can immediately see that the time dependence of $\langle\hat{x}\rangle$ and $\langle\hat{p}\rangle$ are equal to zero since we get $\left\langle n,\, t \mid \hat{x} \mid n,\, t\right\rangle = \left\langle n \,\middle|\, e^{i\left(n+\frac{1}{2}\right)\omega\,t}\,\hat{x}\,e^{-i\left(n+\frac{1}{2}\right)\omega\,t} \,\middle|\, n\right\rangle = \left\langle n \mid \hat{x} \mid n\right\rangle = 0$ from before and similarly for $\langle\hat{p}\rangle$

Its more interesting if we look at the time dependence of the expectation values for a mixture of states, e.g.

$|\alpha\rangle = c_n\,|n\rangle + c_m\,|m\rangle$ where we get for the time dependence $|\alpha,\, t\rangle = c_n\,e^{-i\left(n+\frac{1}{2}\right)\omega\,t}\,|n\rangle + c_m\,e^{-i\left(m+\frac{1}{2}\right)\omega\,t}\,|m\rangle$

$\langle\hat{x}\rangle_{\alpha,t} = \left\langle \alpha,\, t \mid \hat{x} \mid \alpha,\, t\right\rangle = (c_n{}^*\left\langle n \,\middle|\, e^{i\left(n+\frac{1}{2}\right)\omega\,t}\right. + c_m{}^*\left\langle m \,\middle|\, e^{i\left(m+\frac{1}{2}\right)\omega\,t}\right.)\hat{x}(c_n\,e^{-i\left(n+\frac{1}{2}\right)\omega\,t}\,|n\rangle + c_m\,e^{-i\left(m+\frac{1}{2}\right)\omega\,t}\,|m\rangle) =$

$=c_n{}^* \, c_m \, e^{i\,(n-m)\,\omega\,t} \, \langle n \, |\hat{x}|m\rangle + c_m{}^* \, c_n \, e^{i\,(m-n)\,\omega\,t} \, \langle m \, |\hat{x}|n\rangle$ which will have non-zero components of n and m differ by 1

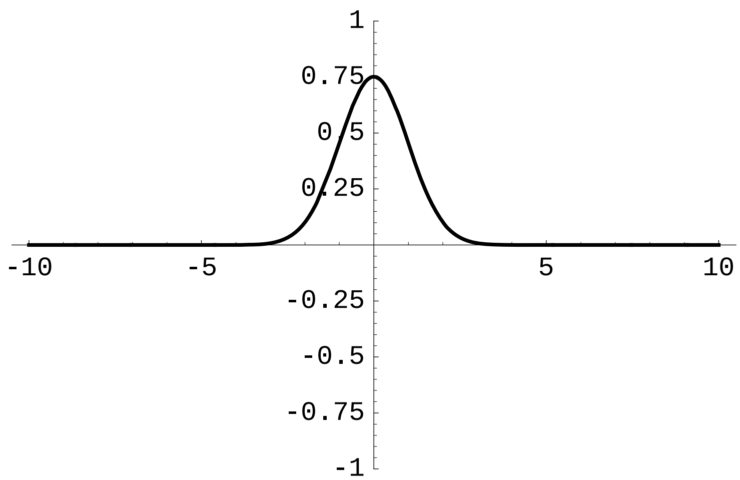
similarly $\langle \hat{p}\rangle_{\alpha,t} = c_n{}^* \, c_m \, e^{i\,(n-m)\,\omega\,t} \, \langle n \, |\hat{p}|m\rangle + c_m{}^* \, c_n \, e^{i\,(m-n)\,\omega\,t} \, \langle m \, |\hat{p}|n\rangle$

Lets take the case where n=0 and m=1 and $|\alpha\rangle = \frac{1}{\sqrt{2}} \, |0\rangle + \frac{1}{\sqrt{2}} \, |1\rangle$

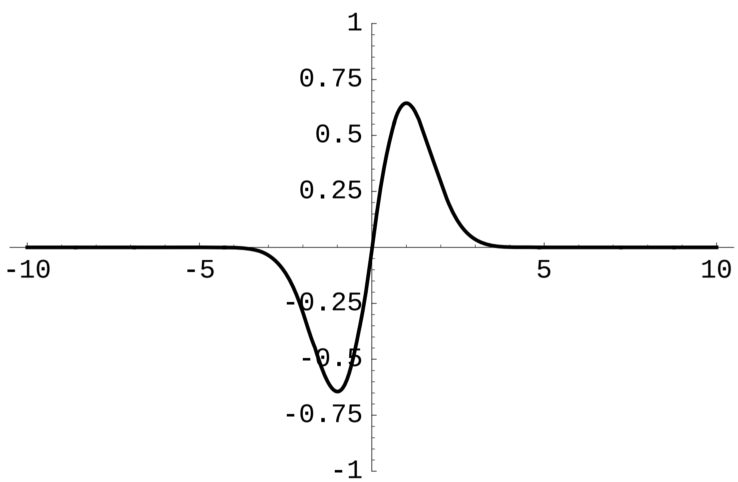
$$\langle \hat{x}\rangle_{\alpha,t} = \frac{1}{2} \sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}} \, e^{-\mathrm{i}\omega\mathrm{t}} \langle 0 \, |\hat{a}^\dagger + \hat{a}|1\rangle + \frac{1}{2} \sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}} \, e^{i\,\omega\,t} \, \langle 1 \, |\hat{a}^\dagger + \hat{a}|0\rangle = \frac{1}{2} \sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}} \, \left(e^{-\mathrm{i}\omega\mathrm{t}} + e^{i\,\omega\,t}\right) = \sqrt{\frac{\hbar}{2\,\mathrm{m}\omega}} \cos\omega\mathrm{t}$$

$$\langle \hat{p}\rangle_{\alpha,t} = \frac{1}{2} \, i \sqrt{\frac{\mathrm{m}\hbar\omega}{2}} \, e^{-\mathrm{i}\omega\mathrm{t}} \langle 0 \, |\hat{a}^\dagger - \hat{a}|1\rangle + \frac{1}{2} \, i \sqrt{\frac{\mathrm{m}\hbar\omega}{2}} \, e^{i\,\omega\,t} \, \langle 1 \, |\hat{a}^\dagger - \hat{a}|0\rangle = \frac{1}{2} \, i \sqrt{\frac{\mathrm{m}\hbar\omega}{2}} \, \left(e^{-\mathrm{i}\omega\mathrm{t}} - e^{i\,\omega\,t}\right) = \frac{1}{2} \sqrt{\frac{\mathrm{m}\hbar\omega}{2}} \sin\omega\mathrm{t}$$

Lets take a look at these graphically, first the n=0
state

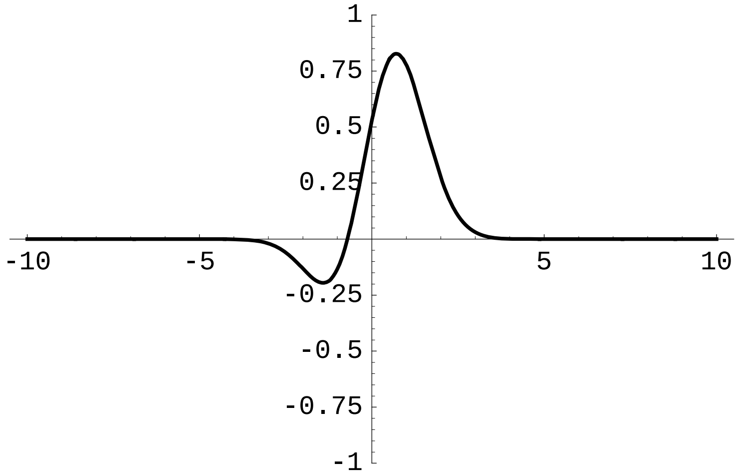


then the n=1 state

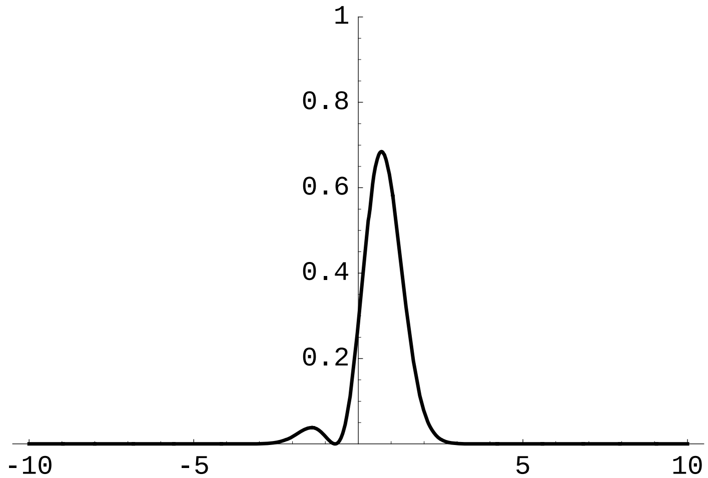


then the sum

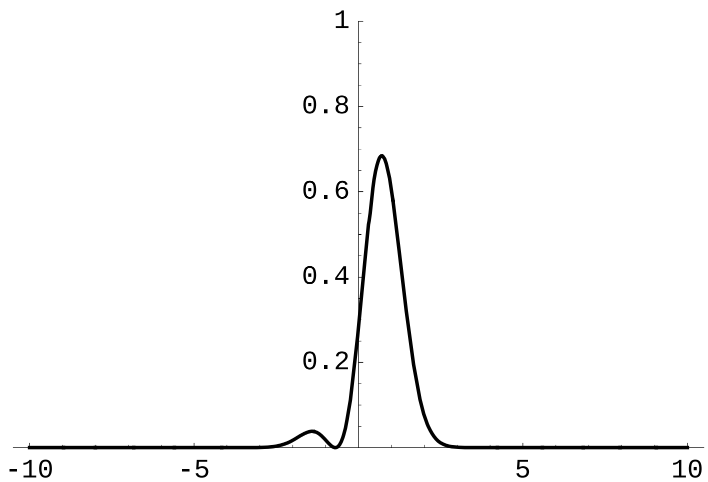
$|\alpha\rangle = \frac{1}{\sqrt{2}} \, |0\rangle + \frac{1}{\sqrt{2}} \, |1\rangle$

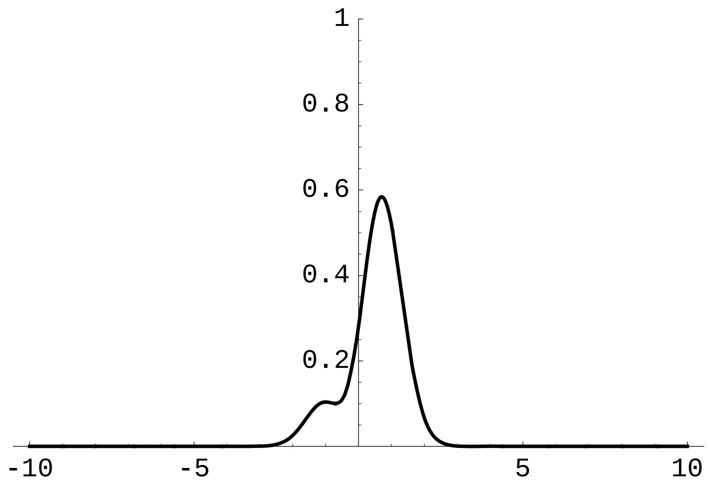


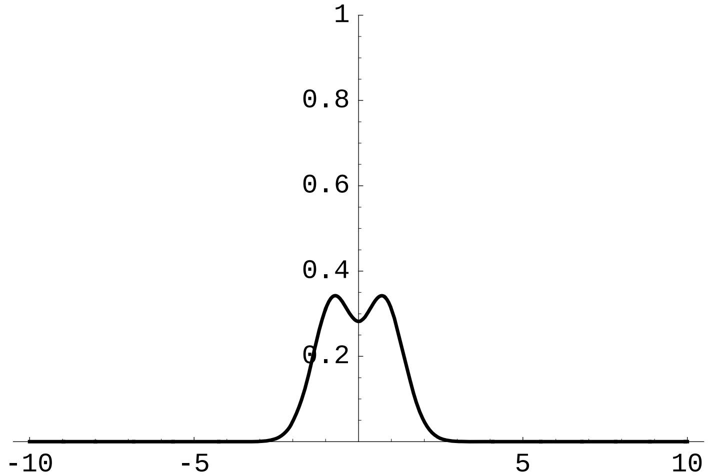
and the corresponding probability



Now here is a series of plots of $\langle \hat{x} \rangle$ where ωt = 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $3\,\frac{\pi}{4}$ and π




1
0.8
0.6
0.4
0.2
-10
-5
5
10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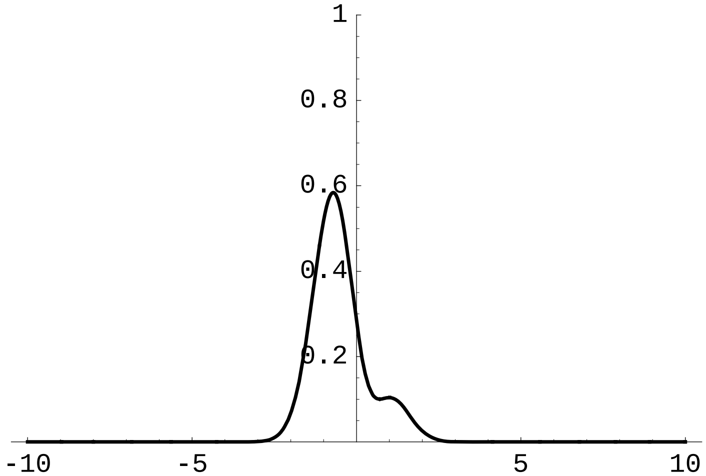
0.6
0.4
0.2
-10
-5
5
10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0.2
-10
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5
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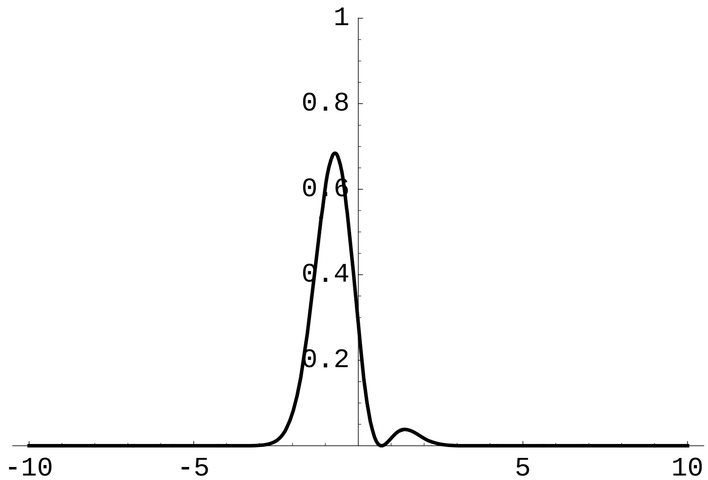
1
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0.6
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10