Appendix: Error Analysis

Introduction:

The study of errors in measurement is an important aspect of all physical sciences. Physics, in particular, is an empirical science; that is, physicists are concerned about deducing physical laws from a collection of data that represent certain physical phenomena. For instance, Newton's second law states that the acceleration, $\boldsymbol{a}$, of an object of mass m is proportional to the *net* force, $\boldsymbol{F}_{net}$, on that object and inversely proportional to its' mass. Newton's second law is an empirical law – it was deduced by Sir Isaac Newton after a repeated number of measurements on objects subject to constant forces. Verification of Newton's laws comes by performing experiments and verifying whether or not Newton's laws are indeed satisfied. An example of this would be conservation of linear momentum ($\boldsymbol{p} = m\ \boldsymbol{v}$, which you will learn about in this course). Newton's first law states that when there is no *net* external force on an object, that object will experience no change in its' momentum. This can be verified by measuring the speed and mass of an object that is subject to no *net* external forces over a period of time and determining whether or not its' momentum remains constant (for a point particle the mass will remain constant, so all we need to do then is to measure the objects velocity over a period of time and determine if it's constant).

In most of our labs we will be concerned with determining the error in length measurements and how to report them. We will also be concerned with how errors are propagated throughout calculations – i.e. what is the error in the volume of a sphere, ΔV, if the uncertainty in its radius is Δr, or what is the uncertainty in the length, ΔL, of a series or connected rods if each rod has an uncertainty of Δl? It is important to note that in this context "error" does not necessarily mean that we've made a mistake in carrying out a measurement, but that we recognize the fact that all physical measurements are imperfect (no physically measured quantity can be determined with infinite precision[1]).

Systematic Errors:

Broadly speaking, there are two different kinds of errors, systematic and statistical (or random) errors. An example of a systematic error might be that our motion sensor is offset by *1 cm*. (which you may come across over the course of the semester) and we are not yet aware of it. If we were then to measure the position of the cart the distribution of data points will all be offset by some amount (in this case *1 cm*.). Other sources of systematic error may be poor measurement technique, instrument calibration, or failures to correct for external conditions. For instance, I want to know the speed of a particular type of ant. If I measure the time it takes for an ant to cross *1 m* of terrain I can calculate the speed by use of the formula $v = x / t$. In this case systematic errors might include things such as; I am consistently *0.1 s* slow in stopping my stopwatch (this will lead to values of speed that are consistently too small), and/or I didn't take into account that to

[1] The term precision (and also accuracy) will be defined later in the introduction.

an ant the distance traveled is greater than *1* meter (there are a lot of "hills" and "valleys" that the ant must traverse, and so the ant may end up traveling a distance of *1.08 m* instead, which will also lead to values of speed that are too small). Systematic errors can be introduced at any time during an experiment, and are sometimes difficult to detect and the sizes difficult to estimate, and so we must consider them at every stage of the experiment (see Taylor chapter 4).

Statistical Errors:

Statistical or random errors are much easier to deal with. An example of a statistical error might be the error introduced when measuring the length of something, a rod say, using a coarse graded ruler. The fineness of the scale readings (how close the lines are on the ruler) is limited and the scale markings have a nonzero width (the lines are thick so how do we know exactly where the end of the rod is?). In either case the length of the rod must be estimated at this point, and is therefore uncertain to a degree. On average we should expect that random errors like this will cancel if we make many measurements, since half the time we may overestimate the length, and half the time we may underestimate it. The best estimate of the standard error associated with the mean (average) value is called "the standard error in the mean" and is calculated by dividing the error, σ_x, divided by the square root of the number of measurements N, that is, $SE(x) = \sigma_x / \sqrt{N}$. This type of error obviously decreases as the number of measurements is increased.

Precision and Accuracy:

The terms precision and accuracy need some attention. The figure below (Fig. 1) should clear up the meanings of each. A student throws a dart at the dartboard a number of times and each landing point is marked with a dot. In the bottom right figure the student is both precise and accurate. We say precise because the spread of the dots (error in the mean) is small – each time the dart lands close to where it was on the last throw, and accurate because the student is hitting the "bull's-eye" (the value we get is close to the "true" value). The figure on the top right displays a student that is precise (dots close together) but not very accurate (they all land off to the upper left). We say that the student who throws darts at the board on the top left is accurate because the average value for the position of the darts on the board is very near the center of the bull's-eye; however this student is not very precise (dart holes are all over the place). Finally, in the figure on the bottom left we see that this student is neither accurate (the mean is off to the left) nor precise. In general, when we quote the uncertainty in an experimentally determined value we are referring to the precision with which that result has been obtained. The accuracy is generally dependant on how well we can eliminate or compensate for systematic errors.

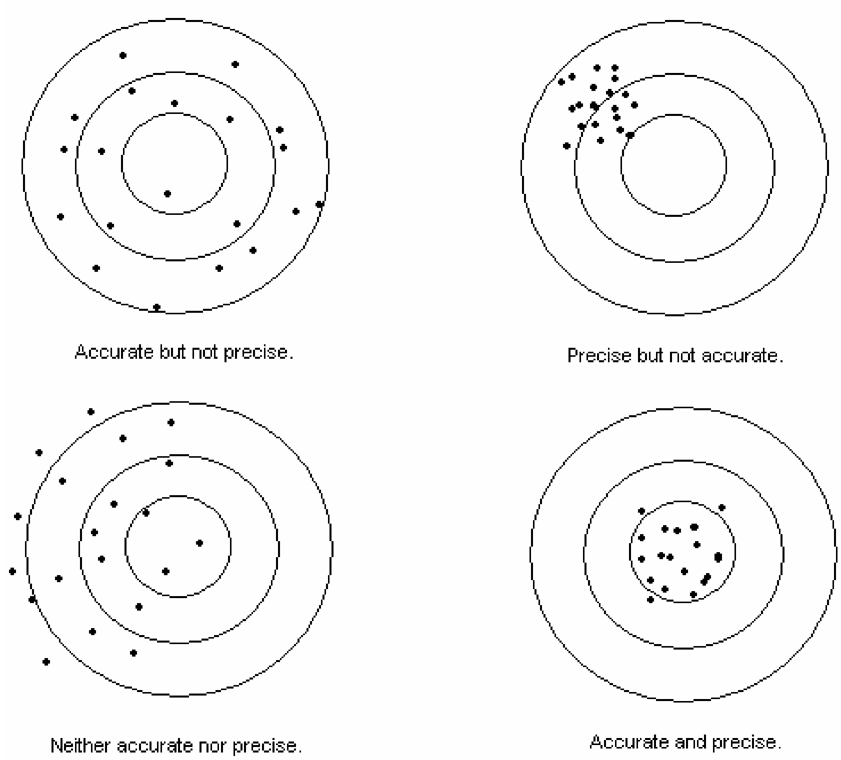

Figure 1: Results of four students' dart tournament. Which one(s) show a sign of systematic errors?

Analysis of Errors:

There are various quantities that we will want to compute in order to accurately convey the results of an experiment. These quantities are the sample mean, standard deviation, and the standard error in the mean. These are calculated as follows

$$\langle X \rangle = \frac{\sum_{i=1}^{N} X_i}{N} \quad , \qquad \sigma_{SD} = \sqrt{\frac{\sum_{i=1}^{N}\left(X_i - \langle X \rangle\right)^2}{N-1}} \quad , \text{ and } \quad SE = \frac{\sigma_{SD}}{\sqrt{N}} \quad ,$$

where X_i is the result of an individual measurement and N is the total number of measurements. When reporting errors we use the form; *measured quantity = <x> ± SE(x)*.

Error Propagation:

One last item that needs our attention is *error propagation*. That is, since most physical measurements do not usually result in the quantity of interest, we must ask how our final result (the dependant variable) is affected due to uncertainties in measured quantities (independent variables). Our task then is to determine how the uncertainty in the dependant variable is calculated. We will not be concerned with how these relationships are derived, but only with the results, which have been reduced to a few simple equations. Namely (where X, Y, and Z are the independent variables),

Addition and subtraction: $f = X \pm Y$

$$\Delta f = \sqrt{\Delta X^2 + \Delta Y^2} \ .$$

Multiplication and division: $f = X\,Y\,Z$ or $f = X\,Y\,/\,Z$ etc…

$$\frac{\Delta C}{C} = \sqrt{\left(\frac{\Delta X}{X}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2 + \left(\frac{\Delta Z}{Z}\right)^2} \ .$$

Powers: $f = X^n$.

$$\frac{\Delta f}{f} = n\frac{\Delta X}{X} .$$

References:

- P. R. Bevington, D. K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences*, McGraw-Hill 1992.
- J.R. Taylor, *An Introduction to Error Analysis*, *2nd Ed.*, University Science Books (Sausalito) 1997.