Notes for Quantum Mechanics

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Updated for 2005

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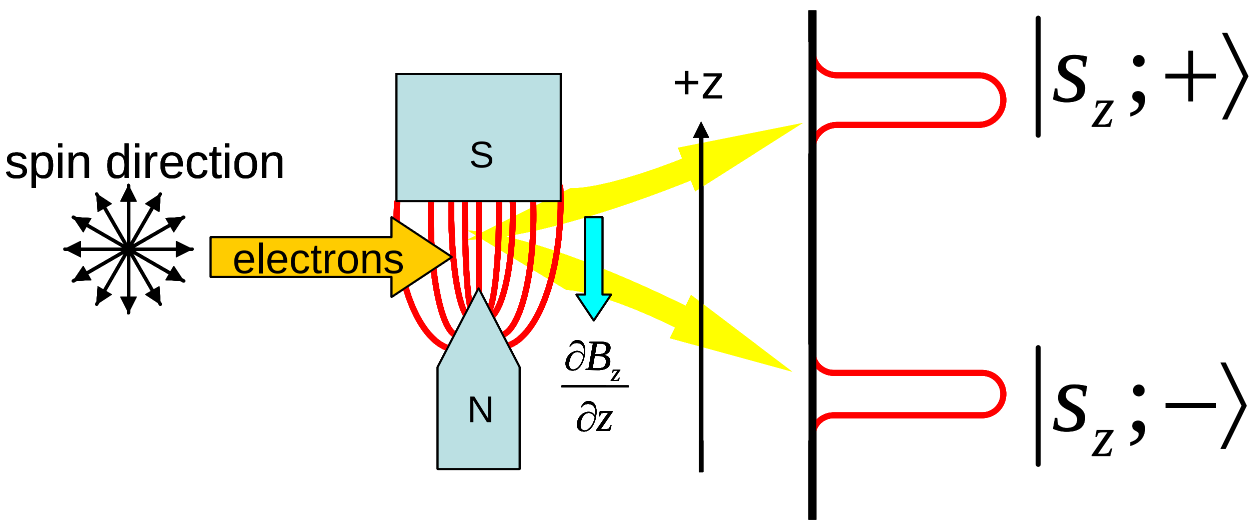
Lecture 8-Stern - Gerlach cont.



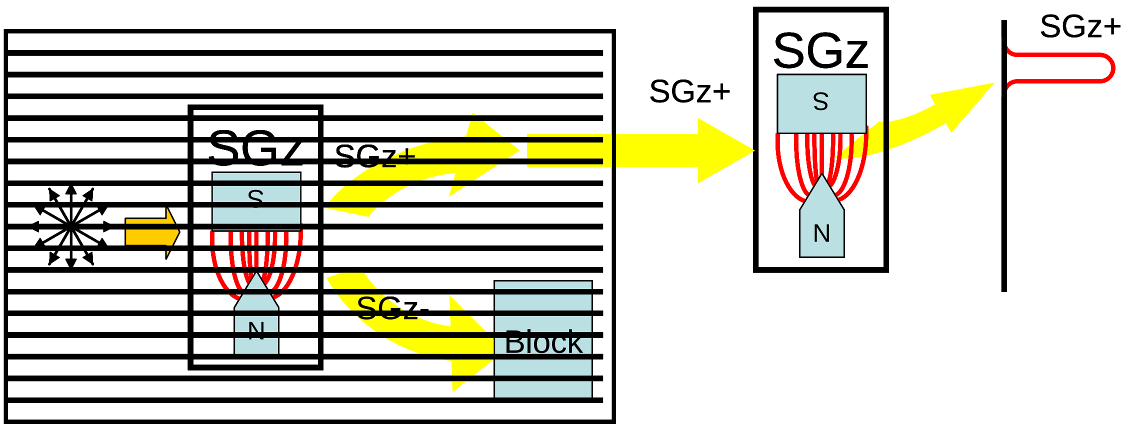
Let's now review this rather strange state of affairs.

1) We start with electron with a magnetic moment which is pointed randomly in some direction. Hence its z component μ_z should have a spread between a maximum $+\mu$ and a minimum $-\mu$

2) We have put it though a magnetic field which deflects the path of the electron in proportion to μ_z and we find that instead of a spread of deflection angles, there are only 2 corresponding to μ_z=either $+\mu$ or $-\mu$ AND NOTHING IN BETWEEN, as if the electron were initially oriented only in the +z or -z direction.

It looks like there are two kinds of μ_z (or $S_z - z$ component of spin). Its also clear that the "spin" we are talking about is funny. Its not really a ball spinning (otherwise it would have a continuous set of z values instead of just 2). We will continue to call it spin - Schwinger said that we borrow the word "spin" from classical physics. It sort of behaves like spin in that it leads to a magnetic moment, but its not the same thing as the classical spin. Let us also remember what we learned about kets and juice. Lets think of S_z like juice. Instead of 3 kinds $-$ there are only 2 $-$ the " + " and " $-$ " types. Now to have been absolutely clear about the juice referring to orange juice was a type of juice we could have written it as $|\text{juice; orangejuics}\rangle$ - that way we remember. As time goes by, we might want to suppress the juice label. (I usually don't refer to myself as name;Richard, but just as Richard. I assume you understand my name is Richard, and that you won't mistake it for some other characteristic of mine - like my profession, or my age etc.) So likewise I am going to write the electrons as one of two states $|S_z;+\rangle$ referring to the electrons that are pushed upward or $|S_z;-\rangle$ (analogous to $|$juice; orangejuice$\rangle$ or $|$juice; grapejuice$\rangle$). So what we have is a 2 dimensional Hilbert space of S_z, just like we had a 3 dimensional Hilbert space of juice.

We will assume that these kets are normalized - i.e. that $\langle S_z;+|S_z;+\rangle=1$ and $\langle S_z;-|S_z;-\rangle=1$. We will also assume that these two states are orthogonal to each other - which makes sense - there is no SGz- component in an SGz+ beam - so $\langle S_z;+|S_z;-\rangle=0$ and $\langle S_z;+|S_z;-\rangle=0$ (watch it - $|S_z;+\rangle$ and $|S_z;-\rangle$ are ORTHOGONAL - unlike $+\hat{z}$ or $-\hat{z}$ vectors which are parallel)

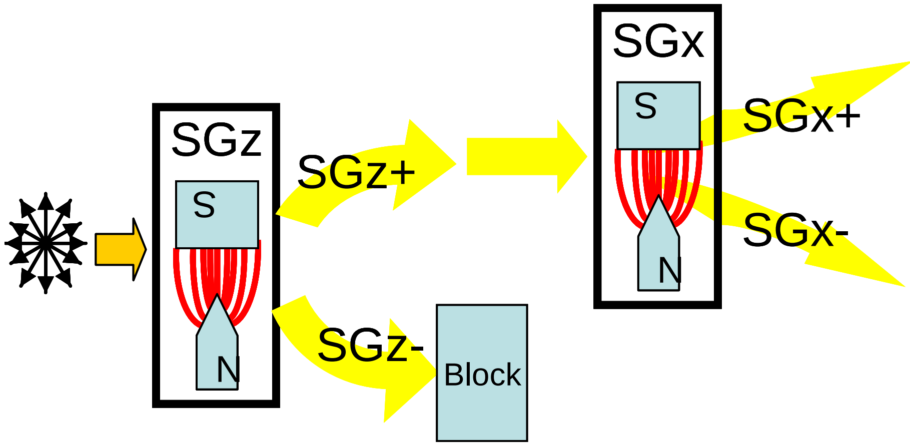


We can think of the SGz apparatus as an operator which operates on the electron and tells us what S_z state it is in (+ or -). If the electron is deflected up, it is telling us it in the + state (similarly for -) We can see this if we look at the situation from before and consider the shaded portion as just a black box which gives us a beam of SGz+ electrons. The second SGz device then acts as an operator which tells us that the beam is in the SGz+ state. We can write this as

$$\hat{S}_z\,|S_z;+\rangle = +\frac{\hbar}{2}\,|S_z;+\rangle \qquad (1)$$

Now why is there an $+\frac{\hbar}{2}$ in front instead of just as +1? It turns out when making measurements the units of this number are angular momentum units and the amount of angular momentum carried by an electron is $\frac{\hbar}{2}$ where $\hbar = 1.05457148 \times 10^{-34}\ \mathrm{m}^2\ \mathrm{kg}\,/\,\mathrm{s}$. We of course have now DEFINED an operator $\hat{S}_z$ which does exactly what the SGz apparatus does.

Now you might pause and think back to what we saw. In fact we could do the exact same thing with SGx or SGy. Which do we use? All of these descriptions of the beam must be equally good. This is similar to the situation in which we can define a vector in one coordinate system - lets call them $\hat{x}$, $\hat{y}$, $\hat{z}$ or you can use another coordinate system $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ (Careful again: $|S_z;+\rangle$ and $|S_z;-\rangle$ is analogous to the x, y, z coordinate system and $|S_x;+\rangle$ and $|S_x;-\rangle$ would be analogous to x' y' z' system) So we can write a vector in either coordinate system. We can also figure out a way to go from one coordinate system or another - we call it a rotation. So is it possible that we might be able to figure out how to write SGz kets in terms of SGx or SGy states?



We get a clue as to how to do this from the experiment where we made an SGz + beam and put it into an SGx apparatus. It split up into SGx+ and an SGx-. So perhaps

$|S_z;+\rangle = \frac{1}{\sqrt{2}}\,(|S_x;+\rangle + |S_x;-\rangle)$. Now when we split an Sz- beam we also get SGx+ and an SGx-, but we cannot make it the same combination since $|S_z;+\rangle$ must be orthogonal to $|S_z;-\rangle$. But there is another combination we can use - (note there was a wrong minus sign in notes till 11/1/05)

$|S_z;-\rangle = \frac{1}{\sqrt{2}}\,(|S_x;+\rangle - |S_x;-\rangle)$.

Now this is still normalized. Lets check and see if the two are orthogonal

$\langle S_z;+|S_z;-\rangle = \frac{1}{\sqrt{2}}\,(\langle S_x;+| + \langle S_x;-|)\,\frac{1}{\sqrt{2}}\,(|S_x;+\rangle - |S_x;-\rangle) = \frac{1}{2}(1+0-0-1)=0$ So it is orthogonal as we wished.

Now remember when we used an SGz+ or SGz- beam and then put it through an SGy device, once again it split in two. So how can we write SGz+ in terms of SGy's? It can't be the same combination because that would mean that $|S_x;+\rangle = |S_y;+\rangle$ and this isn't true. We could test this by using an SGx+ beam and putting it through an SGy device. In fact, we would get the beam splitting into both SGy+ and SGy-.

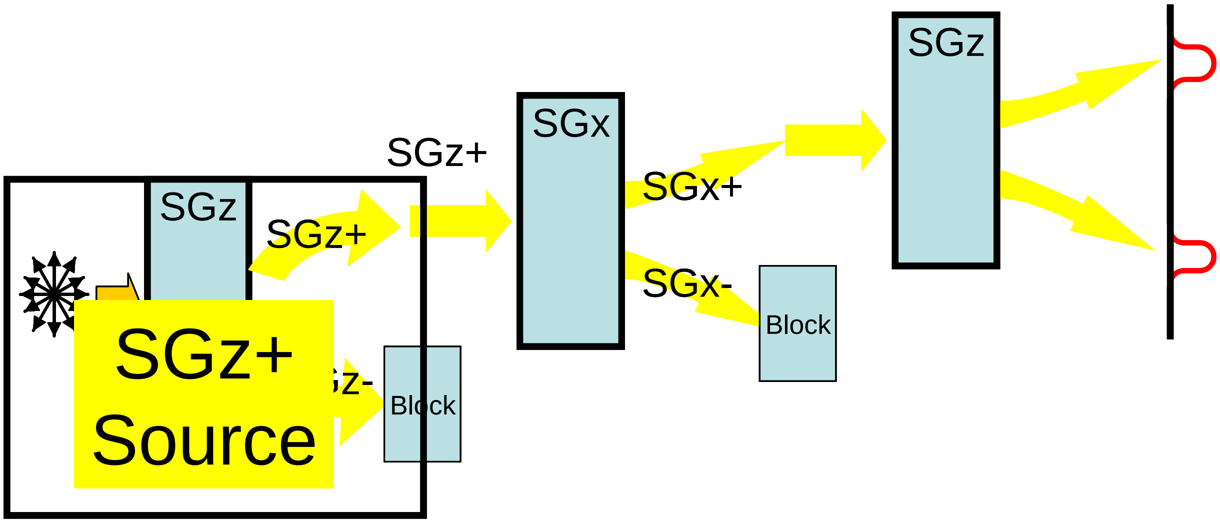
So it there any other combination of kets we could use? There is - remember we can use complex numbers! We will use

$|S_z;+\rangle = \frac{1}{\sqrt{2}}\,(|S_y;+\rangle + i\,|S_y;-\rangle)$.

and $|S_z;-\rangle = \frac{1}{\sqrt{2}}\,(|S_y;+\rangle - i\,|S_y;-\rangle)$.

Again if you check, you will see that these are orthogonal.

Now we can make a check of our formulae by looking at the following situation where we regenerate the SGz- component from an SGz+ beam by first putting it through a SGx apparatus.



We consider the stuff in the box as a source of SGz+ electrons, and normalize it right after the source. So coming out of the source we have

$|S_z;+\rangle$ which we can write as

$|S_z;+\rangle = \frac{1}{\sqrt{2}}(|S_x;+\rangle + |S_x;-\rangle)$

Now passing through the SGx apparatus we block off the SGx- component and we are left with

$\frac{1}{\sqrt{2}}|S_x;+\rangle$ where we have not renormalized it. If we ask the magnitude of this thing we just get $\frac{1}{2}$ which makes sense.

Now we turn the equations for SGz+ and SGz- in terms of the SGx's around to get

$|S_x;+\rangle = \frac{1}{\sqrt{2}}(|S_z;+\rangle + |S_z;-\rangle)$ so we can then substitute this in and we get that

$\frac{1}{\sqrt{2}}|S_x;+\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|S_z;+\rangle + |S_z;-\rangle)$ - this is the beam going into the second SGz device. We can then ask what the final SGz- component is as we get

$\frac{1}{2}|S_z;-\rangle$

Now if we ask the magnitude we get $\frac{1}{4}$. So it looks like our formalism allows us to regenerate the SGz- component, but is magnitude is $\frac{1}{4}$ of the beginning, which of course matches the experiment.

So lets now summarize what we have. First lets start dropping the S_z label in the kets.We will keep the S_x and S_y. We then have

$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$ (where the S_z should be understood in the kets without *a* label) (2)

$|S_y;\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm i\,|-\rangle)$ (3)

we also had (4)

$$\hat{S}_z\,|S_z;+\rangle = +\frac{\hbar}{2}\,|S_z;+\rangle \tag{5}$$

$$\hat{S}_z\,|S_z;-\rangle = -\frac{\hbar}{2}\,|S_z;-\rangle \tag{6}$$

Some inverse relationships

$$|+\rangle = \frac{1}{\sqrt{2}}(|S_x+\rangle + |S_x-\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|S_x+\rangle - |S_x-\rangle) \tag{7}$$

$$|+\rangle = \frac{1}{\sqrt{2}}(|S_y+\rangle + |S_y-\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}\,i}(|S_y+\rangle - |S_y-\rangle) \tag{8}$$

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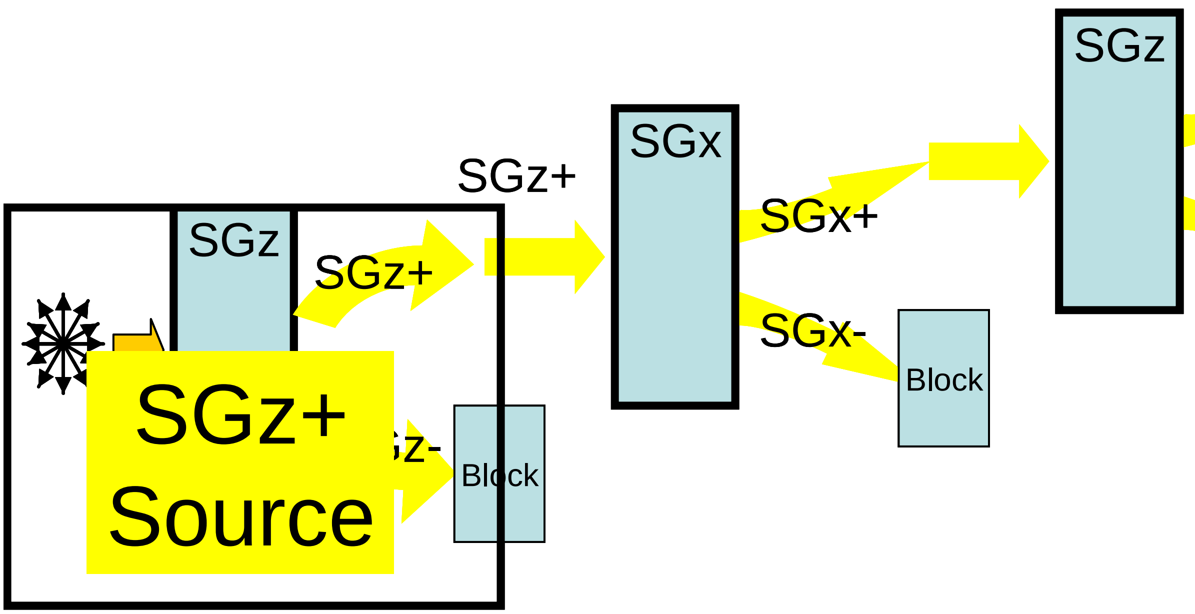

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