Notes for Quantum Mechanics

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Lecture 23
3D, Rotations, and Angular momentum

We will be spending the rest of the quarter on rotations and angular momentum in 3 dimensions. This will lead to the hydrogen atom once we add a $1/r$ potential. We had a pattern of getting to these operators. The idea of translations lead us to momentum, time evolution lead us to energy. We saw that the homogeneity of space and time led to momentum and energy conservation. We will now consider rotations, from which we will get angular momentum. This will not be just the angular momentum of stuff going around in circles however. [Don't worry - it will include that]. This way of deriving the idea of angular momentum will force us to see that there is such a thing as spin which carries an angular momentum even though nothing is going around. It will also force us to give it a value of $\frac{\hbar}{2}$ –Rather the value $\hbar$ is from experimental measurements, but the $\frac{1}{2}$ will be forced on us. We will also see some very funny things about the rotational properties of spin $\frac{1}{2}$ objects.

Lets start by thinking about classical rotations in 3-space. We will use matrix notation and look at the rotation of a vector

$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix}$=(R)$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$ with $RR^T = R^T\,T = 1$ i.e. orthogonal matrices $R^T = R^{-1}$ which means that the norm is preserved i.e. $\sqrt{V_x{}^2 + V_y{}^2 + V_z{}^2}=\sqrt{V'_x{}^2 + V'_y{}^2 + V'_z{}^2}$

The form of a rotation around the z axis by an amount ϕ is $R_z(\phi)=\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we will use the RHR to specify the positive direction. We can expand cos and sin for small rotations ϵ and get

$R_z(\epsilon)=\begin{pmatrix} 1-\frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1-\frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where we have ignored higher order terms in ϵ. We can also write down R_x and R_y

$R_x(\epsilon)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1-\frac{\epsilon^2}{2} \end{pmatrix}$ $R_y(\epsilon)=\begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & -\epsilon \\ 0 & 1 & 0 \\ \epsilon & 0 & 1-\frac{\epsilon^2}{2} \end{pmatrix}$ Lets now take a look at the commutator of R_x and R_y (this is classical!)

$R_x\, R_y = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & 0 & \epsilon \\ \epsilon^2 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix}$ $\quad R_y\, R_x = \begin{pmatrix} 1-\frac{\epsilon^2}{2} & \epsilon^2 & \epsilon \\ 0 & 1-\frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix}$

$[R_x(\epsilon),\ R_y(\epsilon)] = \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R_z(\epsilon^2)-1$ dropping terms of order ϵ^3 This makes sense. If we set ϵ=0, the left gives zero, and for the RHS $R_z(0^2)$=1

This then tells us that rotations do not commute even in classical physics. Try it with a book. Rotate in 90$^\diamond$ around the x axis then 90$^\diamond$ around the y axis. You will NOT get the book in the same position if you rotate first around y, then x.

Aside: For those who are into math, the rotations R form a group, which have an identity operator (1), closure, inverses and associativity.

Now what happens to a ket when you rotate in 3 space? It may be funny. Think of those pictures where it looks like one thing when you look at it from one angle and another when you look at it from a different angle. [These are called Lenticular images] The standard rotation matrix will not tell you that the scene changes. In the same way kets may change in some unexpected way. Lets try to figure this out.

We will do this by analogy with translations in space and time where we figured out what the appropriate operator was. Lets start with a ket $|\alpha\rangle$ and rotate it so $|\alpha\rangle_R = \hat{D}(\mathrm{R})|\alpha\rangle$ where the dimensionality of $\hat{D}$ is that of the ket space, e.g. for spin, it is two dimensions so $\hat{D}$ would be a 2x2 matrix corresponding to the change in the ket space when a rotation is made in 3-D space. (got it?) Its like an operator that tells you how the picture changes when you rotate it in 3-D space. Now let us think of infinitessimal rotation so we can write $\hat{D}(\mathrm{R}(\epsilon)) = 1 - i\hat{G}\epsilon$ where $\hat{G}$ is some operator. This is just like we did for space translations where $\hat{G} = \frac{\hat{p}_x}{\hbar}$ for translations in x and time translations where $\hat{G} = \frac{\hat{H}}{\hbar}$. In this case we will call it $\hat{J}_k$ where k=x,y,z for rotations around the x, y, z axis. So we have $\hat{G} = \frac{\hat{J}_k}{\hbar}$ and $\epsilon = d\phi$ giving

$\hat{D}(\mathrm{R}(d\phi)) = \hat{D}(\vec{n}, d\phi) = 1 - i(\frac{\vec{\hat{J}}\cdot\hat{n}}{\hbar})d\phi$ where $\vec{n}$ is a unit vector which is the axis around which we rotate. NOTE that $\vec{\hat{J}} \neq \vec{\hat{r}} \times \vec{\hat{p}}$ (We will see this come up later and we will define $\vec{\hat{L}} = \vec{\hat{r}} \times \vec{\hat{p}}$ where L is a type of angular momentum called the orbital angular momentum)

For finite rotations $\hat{D}_z = \lim_{N\to\infty}\left[1 - i\left(\frac{\hat{J}_z}{\hbar}\right)\left(\frac{\phi}{N}\right)\right]^N = \exp(-\frac{i\hat{J}_z\phi}{\hbar}) = 1 - \frac{i\hat{J}_z\phi}{\hbar} - \frac{\hat{J}_z^2\phi^2}{2\hbar^2}$ and $\hat{D}(\vec{n}, \phi) = \exp(-\frac{i\vec{\hat{J}}\cdot\hat{n}\phi}{\hbar})$

Just like we did before we will start with a classical formula $[R_x,\ R_y] = R_z - 1$ and assume that these become operators which operate on ket space, i.e. the D's, (actually we can say the D's have the same group properties as the R's) so we will assume $[\hat{D}_x(\epsilon),\ \hat{D}_y(\epsilon)] = \hat{D}_z(\epsilon^2) - 1$ (note that the term on the right has an ϵ^2). We will stick with infinitessimal rotations and we get for the J's (keeping things to order ϵ^2)

$[\hat{D}_x, \hat{D}_y] = \hat{D}_z - 1 \Longrightarrow (1 - \frac{i\hat{J}_x\epsilon}{\hbar} - \frac{\hat{J}_x^2\epsilon^2}{2\hbar^2})(1 - \frac{i\hat{J}_y\epsilon}{\hbar} - \frac{\hat{J}_y^2\epsilon^2}{2\hbar^2}) - (1 - \frac{i\hat{J}_y\epsilon}{\hbar} - \frac{\hat{J}_y^2\epsilon^2}{2\hbar^2})(1 - \frac{i\hat{J}_x\epsilon}{\hbar} - \frac{\hat{J}_x^2\epsilon^2}{2\hbar^2}) = (1 - \frac{i\hat{J}_z\epsilon^2}{\hbar}) - 1$

$\Longrightarrow [1 - \frac{i\hat{J}_y\epsilon}{\hbar} - \frac{\hat{J}_y^2\epsilon^2}{2\hbar^2} - \frac{i\hat{J}_x\epsilon}{\hbar} - \frac{\hat{J}_x\epsilon}{\hbar}\frac{\hat{J}_y\epsilon}{\hbar}\frac{\hat{J}_x^2\epsilon^2}{2\hbar^2}] - [1 - \frac{i\hat{J}_x\epsilon}{\hbar} - \frac{\hat{J}_x^2\epsilon^2}{2\hbar^2} - \frac{i\hat{J}_y\epsilon}{\hbar} - \frac{\hat{J}_y\epsilon}{\hbar}\frac{\hat{J}_x\epsilon}{\hbar} - \frac{\hat{J}_y^2\epsilon^2}{2\hbar^2}] = -\frac{i\hat{J}_z\epsilon^2}{\hbar}$

$\Longrightarrow [-\frac{\hat{J}_x\epsilon}{\hbar}\frac{\hat{J}_y\epsilon}{\hbar}] - [-\frac{\hat{J}_y\epsilon}{\hbar}\frac{\hat{J}_x\epsilon}{\hbar}] = -\frac{i\hat{J}_z\epsilon^2}{\hbar} \quad \Longrightarrow [\hat{J}_x\hat{J}_y] - [\hat{J}_y\hat{J}_x] = i\hbar\hat{J}_z$

$\Longrightarrow [\hat{J}_x, \hat{J}_y] = i\hbar\mathrm{J}_z$ and doing it for the other axis we get

$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$ Very important! Some would go so far as to say this is the eqn that defines angular momentum

At the moment it may seem somewhat abstract (and it is). We will see that angular momentum comes in two flavors. The first is spin, which we know well, and orbital angular momentum. I will show you later how these naturally come about. (I hope you will not be surprised when we use raising and lowering operators again). First I would like to take spin as an example to give you a feel for it. We will see that it has bizarre feature about it.

Spin as an example of angular momentum

As a reminder we had $[\hat{S}_i, \hat{S}_j]=i\hbar\epsilon_{ijk}\hat{S}_k$ which better be true since it is a kind of angular momentum. Now in this case $\hat{D}_z(\phi)=\exp(-\frac{i\hat{S}_z\phi}{\hbar})$ and so we have $\hat{D}_z(\phi)|+\rangle=\exp(-\frac{i\hat{S}_z\phi}{\hbar})\,|+\rangle = e^{-\frac{i\phi}{2}}\,|+\rangle$ and $\hat{D}_z(\phi)|-\rangle = e^{\frac{i\phi}{2}}\,|-\rangle$. Remember that in ket space we have 2 dimensions (+ and -) and in regular cartesian space we have 3-D (x,y,z)

Now lets look at what happens to some ket $|\alpha\rangle$ when we rotate it $|\alpha\rangle_R=\hat{D}(\mathrm{R})|\alpha\rangle$. First lets write $|\alpha\rangle = c_+\,|+\rangle + c_-|-\rangle$

$|\alpha\rangle_R = \hat{D}_z(\phi)\,|\alpha\rangle = \hat{D}_z(\phi)[c_+\,|+\rangle + c_-\,|-\rangle] = c_+ e^{-\frac{i\phi}{2}}\,|+\rangle + c_- e^{\frac{i\phi}{2}}\,|-\rangle$. The interpretation of this ket is not so obvious, but this is what the rotated ket looks like, just as its not so obvious in our picture that it changes when we rotate it.

Instead of figuring out what happens directly to $|\alpha\rangle$ lets look at the expectation value of some spin operator, say $\hat{S}_x$

$\langle\hat{S}_x\rangle=\langle\alpha\,|\,\hat{S}_x\,|\,\alpha\rangle$ and when we rotate the ket we get

$\langle\hat{S}_x\rangle_R = {}_R\langle\alpha\,|\,\hat{S}_x\,|\,\alpha\rangle_R=\langle\alpha\,|\,\hat{D}_z^{\dagger}\,\hat{S}_x\,\hat{D}_z\,|\,\alpha\rangle$. We can do this rather simply by remembering that we can write

$\hat{S}_x = \frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|)$ and $\hat{S}_y = \frac{-i\hbar}{2}(|+\rangle\langle-| - |-\rangle\langle+|)$ so

$\langle\hat{S}_x\rangle_R = \langle\alpha\,|\,\hat{D}_z^{\dagger}\,\hat{S}_x\,\hat{D}_z\,|\,\alpha\rangle = \frac{\hbar}{2}\langle\alpha\,|\exp\left(\frac{i\hat{S}_z\phi}{\hbar}\right)[|+\rangle\langle-| + |-\rangle\langle+|]\exp\left(\frac{-i\hat{S}_z\phi}{\hbar}\right)|\,\alpha\rangle$

$= \frac{\hbar}{2}\langle\alpha\,|\,\{e^{\frac{i\phi}{2}}\,|+\rangle\langle-|\,e^{\frac{i\phi}{2}} + e^{-\frac{i\phi}{2}}\,|-\rangle\langle+|\,e^{-\frac{i\phi}{2}}\}\,|\,\alpha\rangle = \frac{\hbar}{2}\langle\alpha\,|\,\{e^{i\phi}\,|+\rangle\langle-| + e^{-i\phi}\,|-\rangle\langle+|\}\,|\,\alpha\rangle =$

$\frac{\hbar}{2}\langle\alpha\,|\,\{(\cos\phi + i\sin\phi)\,|+\rangle\langle-| + (\cos\phi - i\sin\phi)\,|-\rangle\langle+|\}\,|\,\alpha\rangle = \frac{\hbar}{2}\langle\alpha\,|\,(|+\rangle\langle-| + |-\rangle\langle+|)\cos\phi + (|+\rangle\langle-| - |-\rangle\langle+|)\,i\sin\phi\,|\,\alpha\rangle$

$=\langle\alpha\,|\,\hat{S}_x\cos\phi - \hat{S}_y\sin\phi\,|\,\alpha\rangle=\langle\hat{S}_x\rangle\cos\phi - \langle\hat{S}_y\rangle\sin\phi$

and there are similar relationships as follows for S_y and S_z so

$\langle\hat{S}_x\rangle_R=\langle\hat{S}_x\rangle\cos\phi - \langle\hat{S}_y\rangle\sin\phi$ $\quad\langle\hat{S}_y\rangle_R=\langle\hat{S}_x\rangle\sin\phi + \langle\hat{S}_y\rangle\cos\phi$ $\quad\langle\hat{S}_z\rangle_R=\langle\hat{S}_z\rangle$

So the expectation values of $\hat{S}$ just rotate. It makes sense.

In fact we can write using the usual cartesian rotation matrices that

$\langle\hat{S}_k\rangle_R = \sum_l R_{kl}(\phi)\langle\hat{S}_l\rangle$ and more generally $\langle\hat{J}_k\rangle_R = \sum_l R_{kl}(\phi)\langle\hat{J}_l\rangle$ so the expectation values of angular momentum behave just like an ordinary 3-vector in 3-D space.

Now lets take a look at a single ket again and see what is happening when we rotate it. You will see one of the more bizarre things about QM and spin. Lets write

$|\alpha\rangle=|+\rangle\langle+|\alpha\rangle+|-\rangle\langle-|\alpha\rangle$ Now we will rotate it

$|\alpha\rangle_R = \hat{D}_z(\phi)|\alpha\rangle=\hat{D}_z(\phi)[|+\rangle\langle+|\alpha\rangle+|-\rangle\langle-|\alpha\rangle]=e^{-\frac{i\phi}{2}}\,|+\rangle\langle+|\alpha\rangle + e^{\frac{i\phi}{2}}\,|-\rangle\langle-|\alpha\rangle$ now lets consider a complete rotation, i.e. $\phi=2\pi$

$|\alpha\rangle_{2\pi} = e^{-i\pi}\,|+\rangle\langle+|\alpha\rangle + e^{i\pi}\,|-\rangle\langle-|\alpha\rangle$ $=[\cos(-\pi)+\sin(-\pi)]|+\rangle\langle+|\alpha\rangle+[\cos(\pi)+\sin(\pi)]\,|-\rangle\langle-|\alpha\rangle = -|+\rangle\langle+|\alpha\rangle - |-\rangle\langle-|\alpha\rangle = -|\alpha\rangle$

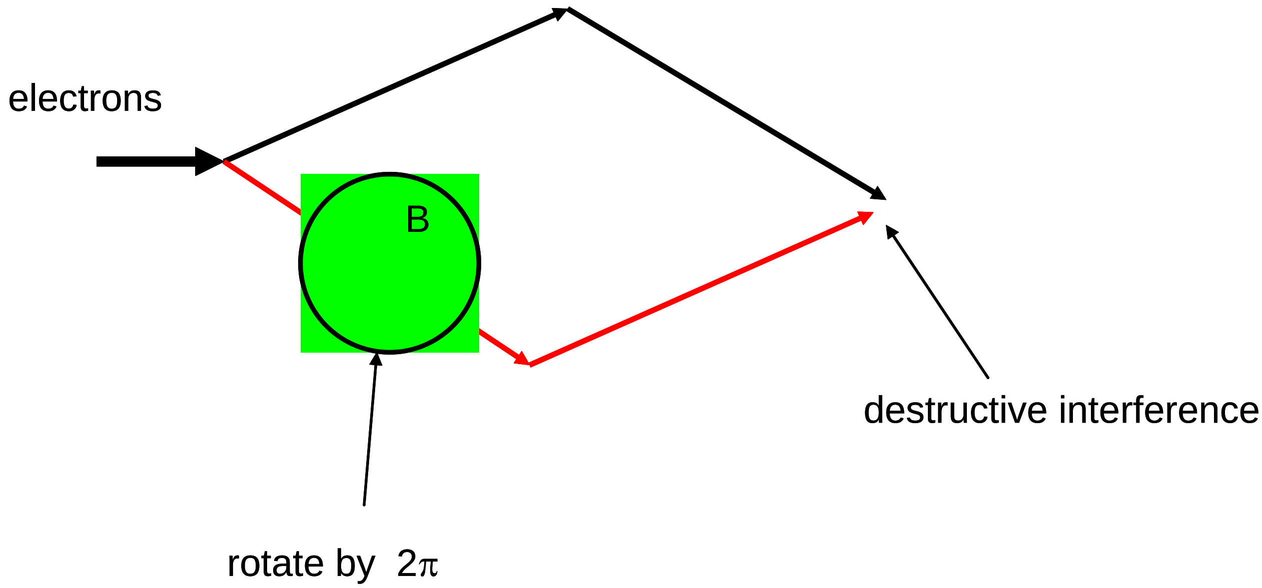
So after a 2π rotation, the ket is NOT the same, but has an extra factor of -1. You have to go around twice before you get back the original!!! $|\alpha\rangle_{4\pi} = |\alpha\rangle$

OK. So you argue that the only think that matters is the square of the wave function so the extra minus sign can't be observed. It turns out, there IS a way to observe it - by using interference.

Remember that you could use magnetic field to rotate spin

$\hat{H} = \vec{\mu} \cdot \vec{B} = -\left(\frac{e}{\text{mc}}\right)\vec{S} \cdot \vec{B}$ (e is negative) Now lets let $\vec{B}$ be a static magnetic field in the z direction so $\hat{H}=\omega \hat{S}_z$ and the time evolution operator is $\exp\left(-\frac{i \hat{H} t}{\hbar}\right)=\exp\left(-\frac{i \omega \hat{S}_z t}{\hbar}\right)$ i.e. it looks exactly like a rotation operator where $\phi=\omega$t

Now you could set up the following apparatus and observe destructive interference after a 2π rotation and constructive interference after 4π. This was actually first done in 1975



Remember Hermonie's warning to Harry Potter about the dangers of going back in time? Strange things happen when folks go back and meet an exact replica of themselves. If they had turned around an odd number of times and met themselves, they would interfere destructively and be annihilated!

Rotations for spin 1 / 2 using matrices

Now we will do rotations for spin 1/2 using matrices. Much of this is just review, but I will add a few definitions. In particular I will introduce χ which are for spin like the wavefunctions ψ(x) i.e. roughly $\langle \pm | \alpha \rangle = \chi$ just like $\langle x | \alpha \rangle = \psi_\alpha(x)$ for the state alpha. Note that in general we will have particles with both kinds of wave functions and will have to specify both of them but lets wait till we get to the hydrogen atom for this.

So $|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle \doteq \begin{pmatrix} \langle + | \alpha \rangle \\ \langle - | \alpha \rangle \end{pmatrix} = \chi$ and $\chi^\dagger = \begin{pmatrix} \langle \alpha | + \rangle & \langle \alpha | - \rangle \end{pmatrix}$

We will also write for the eigenkents of S_z: $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\chi_+{}^\dagger = \begin{pmatrix} 1 & 0 \end{pmatrix}$ $\chi_-{}^\dagger = \begin{pmatrix} 0 & 1 \end{pmatrix}$

So $\chi = \begin{pmatrix} \langle + | \alpha \rangle \\ \langle - | \alpha \rangle \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = c_+ \chi_+ + c_- \chi_-$ $\chi^\dagger = \begin{pmatrix} \langle \alpha | + \rangle & \langle \alpha | - \rangle \end{pmatrix} = \begin{pmatrix} c_+{}^* & c_-{}^* \end{pmatrix} = c_+{}^* \chi_+{}^\dagger + c_-{}^* \chi_-{}^\dagger$

We also have the pauli spin matrices $\hat{S}_k = \frac{\hbar}{2} \sigma_k$ $\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\langle \hat{S}_k \rangle = \frac{\hbar}{2} \chi^\dagger \sigma_k \chi$

Some useful identities

$\{\sigma_i, \sigma_j\} = 2\,\delta_{ij}$ where $\{\hat{a}, \hat{b}\} = \hat{a}\,\hat{b}+\hat{b}\hat{a}$ and $[\sigma_i,\ \sigma_j] = 2\,\mathrm{i}\epsilon_{ijk}\,\sigma_k$

$\sigma_i\,\sigma_j = \delta_{ij} + \mathrm{i}\epsilon_{ijk}\,\sigma_k$ e.g. $\sigma_1\,\sigma_2 = i\,\sigma_3$

$\sigma_k{}^{\dagger} = \sigma_k$ (its hermitian)

$\det(\sigma_i)= -1$

$\mathrm{Tr}(\sigma_i) = 0$

For vectors $\vec{a}$ and $\vec{b}$ $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b})= \vec{a}\cdot\vec{b}+\mathrm{i}\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

Proof::

$$\sum a_j\,\sigma_j \sum b_k\,\sigma_k = \sum_{j,k} \sigma_j\,\sigma_k\,a_j\,b_k = \sum_{j,k} \left(\frac{1}{2}\{\sigma_j, \sigma_k\} + \frac{1}{2}[\sigma_j, \sigma_k]\right) a_j\,b_k = \sum_{j,k} (\delta_{jk} + \mathrm{i}\epsilon_{jkl}\,\sigma_l) a_j\,b_k = \vec{a}\cdot\vec{b}+\mathrm{i}\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

If all components of $\vec{a}$ are real then $(\vec{\sigma}\cdot\vec{a})^2 = |\vec{a}|^2$

Proof: $\vec{\sigma}\cdot\vec{a}=\sum a_k\,\sigma_k = \begin{pmatrix} a_3 & a_1 - \mathrm{i}a_2 \\ a_1 + \mathrm{i}a_2 & -a_3 \end{pmatrix}$

$$(\vec{\sigma}\cdot\vec{a})^2 = \begin{pmatrix} a_3 & a_1 - \mathrm{i}a_2 \\ a_1 + \mathrm{i}a_2 & -a_3 \end{pmatrix}\begin{pmatrix} a_3 & a_1 - \mathrm{i}a_2 \\ a_1 + \mathrm{i}a_2 & -a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(a_1{}^2 + a_2{}^2 + a_2{}^2) = |\vec{a}|^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Rotation operator is $\hat{D}(\vec{n}, \phi)=\exp\left(\frac{-i\,\vec{S}\cdot\hat{n}\,\phi}{\hbar}\right)=\exp\left(\frac{-i\,\vec{\sigma}\cdot\hat{n}\,\phi}{2}\right)$

Now $(\vec{\sigma}\cdot\hat{n})^n=\begin{cases} 1 & n \text{ even} \\ \vec{\sigma}\cdot\hat{n} & n \text{ odd} \end{cases}$

$$\exp\left(\frac{-i\,\vec{\sigma}\cdot\hat{n}\,\phi}{2}\right)= \left[1 - \frac{(\vec{\sigma}\cdot\hat{n})^2}{2!}\left(\frac{\phi}{2}\right)^2 + \frac{(\vec{\sigma}\cdot\hat{n})^4}{4!}\left(\frac{\phi}{2}\right)^4 \ldots\right] + i\left[\frac{(\vec{\sigma}\cdot\hat{n})}{1!}\left(\frac{\phi}{2}\right) - \frac{(\vec{\sigma}\cdot\hat{n})^3}{3!}\left(\frac{\phi}{2}\right)^3 ..\right] = \hat{1}\cos\left(\frac{\phi}{2}\right)\text{-}\mathrm{i}\vec{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right)$$

$=\begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - \mathrm{i}n_z \sin\left(\frac{\phi}{2}\right) & (-\mathrm{i}n_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-\mathrm{i}n_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + \mathrm{i}n_z \sin\left(\frac{\phi}{2}\right) \end{pmatrix}$ This is a handy thing to remember when rotating spin since

$\chi \longrightarrow \exp\left(\frac{-i\,\vec{\sigma}\cdot\hat{n}\,\phi}{2}\right)\chi$

Now since $\langle\hat{S}_k\rangle \longrightarrow \sum_l R_{kl}(\phi)\langle\hat{S}_l\rangle$ and $\langle\hat{S}_k\rangle = \frac{\hbar}{2}\chi^{\dagger}\sigma_k\chi$ we can write $\chi^{\dagger}\sigma_k\chi \longrightarrow \sum_l R_{kl}(\phi)\chi^{\dagger}\sigma_l\chi$

Also note that $\exp\left(\frac{-i\,\vec{\sigma}\cdot\hat{n}\,\phi}{2}\right)=\hat{1}\cos\left(\frac{\phi}{2}\right)\text{-}\mathrm{i}\vec{\sigma}\cdot\hat{n}\sin\left(\frac{\phi}{2}\right) = \begin{cases} -1 & \phi = 2\pi \\ 1 & \phi = 4\pi \end{cases}$