Notes for Quantum Mechanics

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Lecture 20

So where are we? We started out with a harmonic oscillator potential $V(x) = \frac{1}{2}\,\mathrm{m}\omega^2 x^2$. We then assumed QM, which means we took the classical hamiltonian and made everything operators. We found the eigenkets and eigenvalues of $\hat{H}$. As typical in quantum mechanics where we have bound states we found that the energies were quantized, i.e. $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ which gave rise to a second funny QM thing - that the lowest energy is not zero. The eigenkets had characteristics which reminded us of the classical oscillator - that is, its probability distributions in position were similar.

Momentum representation

We have been mostly working in position basis, i.e. looking at things like $\langle x'|\alpha\rangle$. Remember that we can also work in momentum representation. We can go back and forth using the following (from lecture 16)

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\left(\frac{ip'x'}{\hbar}\right)} \quad \text{and} \quad \langle p'|x'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\left(\frac{-ip'x'}{\hbar}\right)}$$

$$\langle x'|\alpha\rangle = \int dp' \langle x'|p'\rangle \langle p'|\alpha\rangle \implies \psi_\alpha(x') = \frac{1}{\sqrt{2\pi\hbar}} \int dp'\, e^{\left(\frac{ip'x'}{\hbar}\right)} \phi_\alpha(p')$$

$$\langle p'|\alpha\rangle = \int dx' \langle p'|x'\rangle \langle x'|\alpha\rangle \implies \phi_\alpha(p') = \frac{1}{\sqrt{2\pi\hbar}} \int dx'\, e^{\left(\frac{-ip'x'}{\hbar}\right)} \psi_\alpha(x')$$

we also had $\langle x'|\,\hat{\boldsymbol{p}}\,|\alpha\rangle = \frac{\hbar}{i}\frac{\boldsymbol{d}}{\mathbf{dx'}}\langle x'|\alpha\rangle$, i.e. a representation of $\hat{\boldsymbol{p}}$ in the position representation.

If we write the time independent Schroedinger eqn for an energy eigenket in the position representation we get

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\langle x'|n\rangle + \frac{1}{2}\,\mathrm{m}\omega^2 x'^2 \langle x'|n\rangle = E\langle x'|n\rangle \quad \text{or} \quad \left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}\,\mathrm{m}\omega^2 x'^2\right)\langle x'|n\rangle = E\langle x'|n\rangle$$

Can we find a representation of $\hat{x}$ in the momentum representation? There is and simillar operator to the translation operator - the momentum boost operator $e^{\frac{ia\hat{x}}{\hbar}}$. We can then follow the logic of lecture 16 and find that

$\langle p'|\,\hat{\boldsymbol{x}}\,|\alpha\rangle = \frac{1}{i\hbar}\frac{\boldsymbol{d}}{\mathbf{dp'}}\langle p'|\alpha\rangle$.

The other way to do this is just to transform the representation.

We know $\langle x'|\,\hat{x}\,|\alpha\rangle = x'\langle x'|\alpha\rangle$ i.e. this is the position operator in position representation. We want $\langle p'|\,\hat{x}\,|\alpha\rangle$. So lets start with $|\alpha\rangle = |x''\rangle$ i.e. $\langle x'|\,\hat{x}\,|x''\rangle = x'\delta(x'-x'')$ and then find $\langle p'|\,\hat{x}\,|p''\rangle$

One important identity we will need from the back of the book is $\delta(k-k') = \frac{1}{2\pi}\int e^{i(k-k')x'}\, dx'$

We can then write $\frac{d}{dk}\delta(k-k') = \frac{1}{2\pi}\int \frac{d}{dk}\, e^{i(k-k')x'}\, dx' = \frac{i}{2\pi}\int x'\, e^{i(k-k')x'}\, dx'$

So lets start

$$\langle p'|\hat{x}|p''\rangle = \int dx' \int dx'' \langle p'|x'\rangle\langle x'|\hat{x}|x''\rangle\langle x''|p''\rangle = \frac{1}{2\pi\hbar}\int dx' \int dx'' e^{-i\frac{p'x'}{\hbar}} x'\,\delta(x'-x'')\,e^{i\frac{p''x''}{\hbar}} =$$

$$\frac{1}{2\pi\hbar}\int dx'\, e^{-i\frac{p'x'}{\hbar}} x'\, e^{i\frac{p''x'}{\hbar}} = \frac{1}{2\pi\hbar}\int dx'\, e^{-i(p'-p'')\frac{x'}{\hbar}} x' = i\hbar \frac{d}{dp'}\delta(p'-p'') = -\frac{\hbar}{i}\frac{d}{dp'}\delta(p'-p'')$$

$$\langle p'|\hat{x}|\alpha\rangle = \int dp'' \langle p'|\hat{x}|p''\rangle\langle p''|\alpha\rangle = -\frac{\hbar}{i}\int dp''\frac{d}{dp'}\delta(p'-p'')\langle p''|\alpha\rangle = -\frac{\hbar}{i}\frac{d}{dp'}\langle p'|\alpha\rangle \quad \text{as before}$$

So in the momentum representation we get

$$\langle p'|\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2|n\rangle = E|n\rangle \;\rightarrow\; \frac{p'^2}{2m}\langle p'|n\rangle + \frac{1}{2}m\omega^2\langle p'|\hat{x}\int dp''|p''\rangle\langle p''|\hat{x}|n\rangle = E\langle p'|n\rangle$$

$$\rightarrow \frac{p'^2}{2m}\langle p'|n\rangle + \frac{1}{2}m\omega^2\int dp''\langle p'|\hat{x}|p''\rangle\langle p''|\hat{x}|n\rangle = E\langle p'|n\rangle$$

$$\rightarrow \frac{p'^2}{2m}\langle p'|n\rangle + \frac{1}{2}m\omega^2\frac{\hbar}{i}\int dp''\langle p'|\hat{x}|p''\rangle\frac{d}{dp''}\langle p''|n\rangle = E\langle p'|n\rangle$$

$$\rightarrow \frac{p'^2}{2m}\langle p'|n\rangle + \frac{1}{2}m\omega^2(-\hbar^2)\int dp''\frac{d}{dp'}\langle p'|p''\rangle\frac{d}{dp''}\langle p''|n\rangle = E\langle p'|n\rangle$$

$$\rightarrow \frac{p'^2}{2m}\langle p'|n\rangle - \frac{1}{2}m\omega^2\hbar^2\int dp''\frac{d}{dp'}\delta(p'-p'')\frac{d}{dp''}\langle p''|n\rangle = E\langle p'|n\rangle$$

$$\rightarrow \frac{p'^2}{2m}\langle p'|n\rangle - \frac{1}{2}m\omega^2\hbar^2\frac{d}{dp'}\frac{d}{dp'}\langle p'|n\rangle = E\langle p'|n\rangle$$

$$\rightarrow \frac{p'^2}{2m}\langle p'|n\rangle - \frac{1}{2}m\omega^2\hbar^2\frac{d^2}{dp'^2}\langle p'|n\rangle = E\langle p'|n\rangle$$

$$\rightarrow \left(\frac{p'^2}{2m} - \frac{1}{2}m\omega^2\hbar^2\frac{d^2}{dp'^2}\right)\langle p'|n\rangle = E\langle p'|n\rangle$$

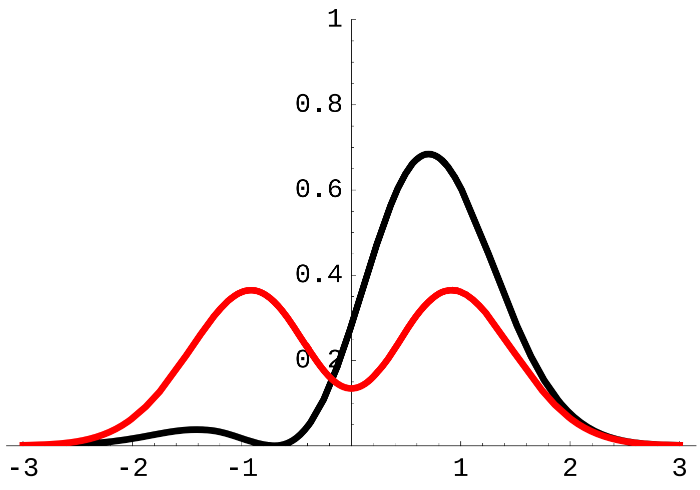
Lets now switch lowest the two energy wave functions to momentum representation. We got

$\langle x|0\rangle = \dfrac{e^{-\frac{x^2}{2}}}{\sqrt[4]{\pi}}$ and $\langle x|1\rangle = \dfrac{\sqrt{2}\,e^{-\frac{x^2}{2}}x}{\sqrt[4]{\pi}}$

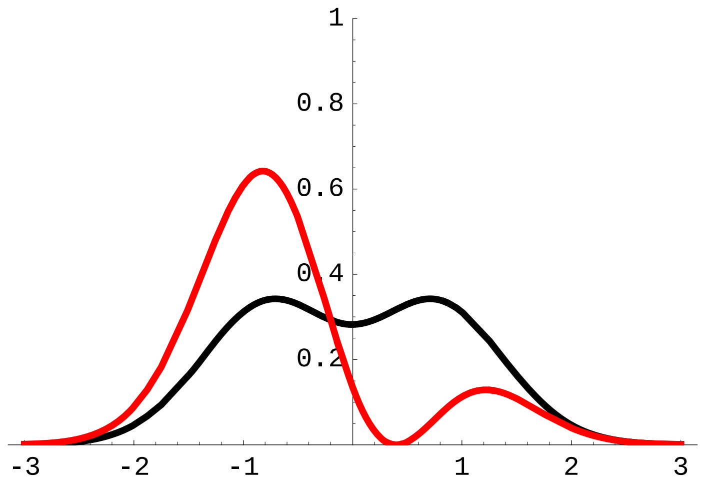
After doing the integrals

$\langle p'|0\rangle = \displaystyle\int dx'\, e^{\left(\frac{-ip'x'}{\hbar}\right)}\frac{e^{-\frac{x'^2}{2}}}{\sqrt[4]{\pi}} = \frac{e^{-\frac{p^2}{2}}}{\sqrt{2}\,\pi^{3/4}}$ $\qquad \langle p'|1\rangle = \displaystyle\int dx'\, e^{\left(\frac{-ip'x'}{\hbar}\right)}\frac{\sqrt{2}\,e^{-\frac{x^2}{2}}x}{\sqrt[4]{\pi}} = -\frac{i\,e^{-\frac{p^2}{2}}p}{\sqrt[4]{\pi}}$

Now these momentum wave functions are the very similar to the position wave functions except for the relative phase of $\langle p'|1\rangle$ and $\langle p'|0\rangle$ which are 90 degrees out of phase with each other - that is, one is pure imaginary while the other is real. This can be seen in the following where we look at our mixture from before - i.e. $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$



You can see this in the probabilities shown here where the black is the position wave function and the red is the momentum wave function. But this makes sense. The black shows the oscillator when its position is at one extreme - that is $\langle\hat{x}\rangle$ is at a maximum. But at the same point we expect the momentum to be zero, classically, and QM we see that $\langle\hat{p}\rangle$=0



We can also see that when $\langle\hat{x}\rangle$ is 0, then $\langle\hat{p}\rangle$ is at a (negative) maximum

This can of course be seen from the eqns we got before

$\langle\hat{x}\rangle_{\alpha,t}=\sqrt{\frac{\hbar}{2\,m\omega}}\cos\omega t$ and $\langle\hat{p}\rangle_{\alpha,t}=\frac{1}{2}\sqrt{\frac{m\hbar\omega}{2}}\sin\omega t$