Physics 156B Quantum Mechanics - Final

March 17, 2005

Exam Solutions

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{2005, 3, 15, 20, 44, 43.8865088}

Total - 120 pts
pr 1 - 20 pts
pr 2 - 20 pts
pr 3 - 20 pts
pr 4 - 20 pts
pr 5 - 20 pts

READ THIS ! ! !
Make sure you do not spend more than 1 / 2 hour on each of these. Don ' t get caught up in long calculations. First explain clearly what steps you will take to solve the problem. That way I can give partial credit even if you do not complete the calculation.
Problems 1, 2 and 5 are straightforward. Problem 3 is easy too but the calculation can get painful if you do not take the smartest route. Don ' t waste time here. Problem 4 has a subtle thing you must recognize, but is otherwise straightforward. I give hints to guide you through the toughest things - these hints assume you know what you are doing. It might be best on these problems to write out a strategy first. That way I can follow your thinking - and more importantly - it forces you to clarify your thinking. I also give you lots of info at the end to save you time of searching around in your notes.

1) Let us assume that there exists a particle like an electron with spin s=$\frac{3}{2}$. Lets call it an setolepton. You manage to replace the electron in an hydrogen atom with a setolepton. Lets assume that it is in a p-state orbital (i.k. l=1 where l is the quantum number for L - the orbital angular momentum)

a) list the j and m_j states that are possible (i.e. quantum numbers of total angular momentum J and the quantum number of J_z)

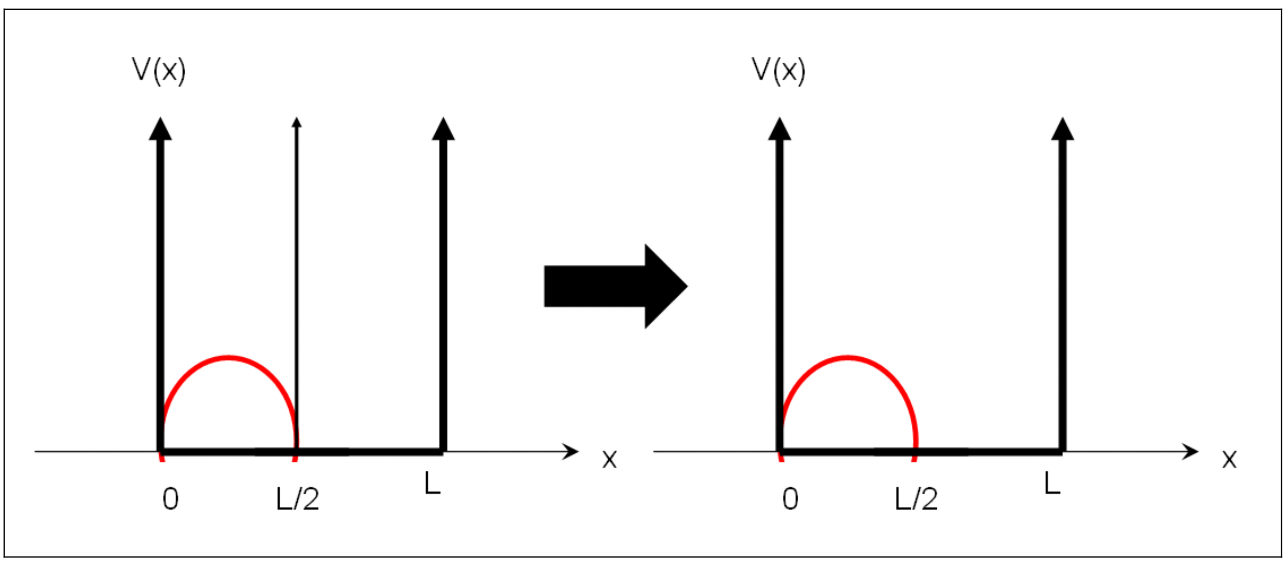
b) Suppose it is in the $j = \frac{3}{2}$, $m_j = \frac{-1}{2}$ state. Write this state in terms of l, m_l, s and m_s. Make sure to be clear about which quantum number stands for what. Caution: check the signs and look at the lower right box of the CG table shown in the exam.

c) To find the correction from spin-orbit coupling we added a term $\hat{H}_{SO} = \frac{e^2}{2m^2c^2r^3}\hat{L}\cdot\hat{S}$. Now as I did in class, to make things easier lets find f=$\langle$ nlsjm $\left| \frac{1}{r^3} \right|$nlsjm$\rangle = \int_0^\infty R^*_{nl}(r)\, \frac{1}{r^3}\, R_{nl}(r)\, r^2\, dr = \frac{1}{l(l+\frac{1}{2})(l+1)\,n^3\,a_0^3}$ and then rewrite $\hat{H}_{SO} = \frac{e^2}{2m^2c^2}\, f\hat{L}\cdot\hat{S}$ where $f = \frac{1}{l(l+\frac{1}{2})(l+1)\,n^3\,a_0^3}$ is a constant. Find the correction to this funny atom for the n=1, l=1, $s = \frac{3}{2}$ $j = \frac{3}{2}$, $m_j = \frac{-1}{2}$ state.

a) We have $\frac{3}{2}\otimes 1$, so $j = \frac{5}{2}$ $m_j = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, \frac{-5}{2}$
so $j = \frac{3}{2}$ $m_j = \frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}$ and so $j = \frac{1}{2}$ $m_j = \frac{1}{2}, \frac{-1}{2}$

b) $|$ l=1 $s = \frac{3}{2}$ $j = \frac{3}{2}$ $m_j = \frac{-1}{2}\rangle$=
$-\sqrt{\frac{8}{15}}\,|$ l=1 $s = \frac{3}{2}$ $m_l = -1$ $m_s = \frac{1}{2}\rangle + \sqrt{\frac{1}{15}}\,|$ l=1 $s = \frac{3}{2}$ $m_l = 0$ $m_s = \frac{-1}{2}\rangle + \sqrt{\frac{2}{5}}\,|$ l=1 $s = \frac{3}{2}$ $m_l = 1$ $m_s = \frac{-3}{2}\rangle$

c) $\langle 1\ \frac{3}{2}\ \frac{3}{2}\ \frac{-1}{2}|\hat{H}_{SO}\left|1\ \frac{3}{2}\ \frac{3}{2}\ \frac{-1}{2}\right\rangle = \frac{e^2}{2m^2c^2}\, f\langle 1\ \frac{3}{2}\ \frac{3}{2}\ \frac{-1}{2}|\hat{L}\cdot\hat{S}|1\ \frac{3}{2}\ \frac{3}{2}\ \frac{-1}{2}\rangle =$
$\frac{e^2}{4m^2c^2}\, f\langle 1\ \frac{3}{2}\ \frac{3}{2}\ \frac{-1}{2}|\hat{J}^2\text{-}\hat{L}^2\text{-}\hat{S}^2|1\ \frac{3}{2}\ \frac{3}{2}\ \frac{-1}{2}\rangle = \frac{e^2}{4m^2c^2}\, \frac{1}{l(l+\frac{1}{2})(l+1)\,n^3\,a_0^3}\, \hbar^2$[j(j+1)-l(l+1)-s(s+1)]=
$\frac{e^2\hbar^2}{4m^2c^2a_0^3}\, \frac{1}{1(1+\frac{1}{2})(1+1)1^3}\left[\frac{3}{2}\left(\frac{3}{2}+1\right) - 1(1+1) - \frac{3}{2}\left(\frac{3}{2}+1\right)\right] = \frac{e^2\hbar^2}{4m^2c^2a_0^3}\, \frac{1}{3}\,(-2) = -\frac{e^2\hbar^2}{6m^2c^2a_0^3} = -\frac{\mathcal{R}\alpha^2}{3}$



2) You are given an infinite potential well as in the left picture above. There is a barrier in the center. A particle is in the left compartment in the n=1 ground state of a particle in a box of size L/2. Suddenly the barrier is removed. (Part a is a big hint to the rest of the problem)

a) i. What is the hamiltonian after the barrier is removed?

ii. What is the first step you should do, given a problem - i.e. this problem, and an observable, i.e. the hamiltonian.

iii. How do you find the time dependence of the n=1 state?

b) Find the time dependence of the state. If you do an expansion, calculate the first 3 non-zero terms. (Hint: write down the initial condition carefully over all of x, not just the left half of the box.)

a) i. $\langle \hat{H} \rangle = \frac{\hat{p}^2}{2m}$ with V=∞ at 0 and 1

ii) find the eigenkets of the hamiltonian then expand the initial condition in terms of these

iii) it is easy then to apply the time evolution operator $e^{-\frac{i \hat{H} t}{\hbar}}$ to the initial condition to find the time dependence of the state

b) The eigenkets of the Hamiltonian are $\varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ with $E_n = \frac{n^2 \hbar^2 \pi^2}{2 \mathrm{mL}^2}$, $n = 1, 2, \ldots$ for $0 < x < L$

The initial condition is $\psi(x, t = 0)$ =A $\sin\left(\frac{\pi x}{L/2}\right)$ = A $\sin\left(\frac{2\pi x}{L}\right)$ from 0 to $\frac{L}{2}$ and 0 between $\frac{L}{2}$ and L. It is unormalized Now we had better normalize it over the full size of the box to fine the normalization constant A.

$\int_0^L | \psi_1(x, t = 0) |^2 \, dx = A^2 \int_0^L \sin^2\left(\frac{2\pi x}{L}\right) dx = A^2 \frac{L}{2} = 1 \quad \to A = \sqrt{\frac{2}{L}}$ so the initial condition is

$\psi_1(x, t = 0) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$ in the interval x=[0 , $\frac{L}{2}$] and =0 in the interval x=[$\frac{L}{2}$, L]

Now lets expand in terms of the eigenkets of $\hat{H}$

$\psi(x, t = 0) = \sqrt{\frac{2}{L}} \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

$c_n = \sqrt{\frac{2}{L}} \int_0^L \psi(x, t = 0) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^{L/2} \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 2 \frac{2 \sin\left(\frac{n\pi}{2}\right)}{4\pi - n^2 \pi}$ Now the n= 2 and 4 terms are zero

so $c_1 = \frac{4}{3\pi}$ $c_3 = \frac{4}{5\pi}$ $c_5 = -\frac{4}{21\pi}$ So we get

$\psi_1(x, t = 0) = \sqrt{\frac{2}{L}} \left[\frac{4}{3\pi} \sin\left(\frac{\pi x}{L}\right) + \frac{4}{5\pi} \sin\left(\frac{3\pi x}{L}\right) - \frac{4}{21\pi} \sin\left(\frac{3\pi x}{L}\right) \ldots\right]$ so we have completed ii.

Now for the time dependence i.e. step

First lets define $\omega = \frac{E_1}{\hbar} = \frac{\hbar \pi^2}{2 \mathrm{mL}^2}$

The time evolution operator is $e^{-\frac{i \hat{H} t}{\hbar}}$ which when operating on some eigenket of $\hat{H}$ adds a factor $e^{-i \frac{n^2 \hbar^2 \pi^2}{2 \mathrm{mL}^2 \hbar} t} = e^{-\mathrm{in}^2 \omega t}$

So the final answer is

$\psi(x, t) = \frac{4}{\pi} \sqrt{\frac{2}{L}} \left[\frac{1}{3} \sin\left(\frac{\pi x}{L}\right) e^{-i\omega t} + \frac{1}{5} \sin\left(\frac{3\pi x}{L}\right) e^{-9 i\omega t} - \frac{1}{21} \sin\left(\frac{3\pi x}{L}\right) e^{-25 i\omega t} \ldots\right]$

Integrate[Sin[2 * Pi * x] * Sin[n * Pi * x], {x, 0, 1 / 2}]

$$\frac{2 \sin\left(\frac{n\pi}{2}\right)}{4\pi - n^2 \pi}$$

3) When we made our simple model of the strong interactions we assumed that the particles were subject to a one dimensional simple harmonic oscillator potential $\frac{1}{2} \mathrm{m}\omega^2 \hat{x}^2$. It was a non-relativistic model. Actually for the lighter quarks, we should use a more realistic relativistic hamiltonian. In this problem we will figure out the first order relativistic correction.

a) we have always used the non -relativistic $p^2/2m$ form for the kinetic energy. The relativistic form should be T=$\sqrt{p^2 c^2 + m^2 c^4}$ $-$mc^2.

i. Show that the first two terms in an expansion of this would be $\frac{p^2}{2m} - \frac{1}{8 m^3 c^2} p^4$.

ii. We will define $\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$ and $\hat{H}' = -\frac{1}{8 m^3 c^2} \hat{p}^4$. What are the energies and eigenlets of H_0?

b) Now find the first order relativistic correction to the energy, i.e. $\langle \hat{H}' \rangle$, were the expectation value is taken between the eigenkets of H_0. (Hint: First write down the procedure clearly and completely that you should follow with the relevant formulae. That is mostly what I am looking for. Then work it out- if you are having pages and pages - then there is a cleverer way to do it. I found it quickest to operate early in the kets.)

a) i. $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) \ldots$

$\sqrt{p^2 c^2 + m^2 c^4} = mc^2 \sqrt{1 + \frac{p^2 c^2}{m^2 c^4}} = mc^2 (1 + \frac{p^2 c^2}{m^2 c^4} \frac{1}{2} - \frac{p^4 c^4}{m^4 c^8} \frac{1}{4} \frac{1}{2} \ldots.)$

T=$\sqrt{p^2 c^2 + m^2 c^4} - mc^2 = \frac{p^2}{2m} - \frac{1}{8 m^3 c^2} p^4$

ii) SHO $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle$ $\quad E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ or you could have written

b) First use $\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}} (\hat{a}^\dagger - \hat{a})$ to write

$\hat{H}' = -\frac{1}{8 m^3 c^2} \left(\frac{m\hbar\omega}{2}\right)^4 (\hat{a}^\dagger - \hat{a})^4$ Then use $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

when finding the corrections i.e. $\langle n | \hat{H}' | n \rangle$. It turns out using commutators works, but it takes longer in which case you should use $[\hat{a}, \hat{a}^\dagger]=1$ and $[\hat{N},\hat{a}] = [\hat{a}^\dagger \hat{a}, \hat{a}] = -\hat{a}$ and $[\hat{N},\hat{a}^\dagger] = [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger$

Now lets do it

$\hat{H}' = -\frac{m}{128 c^2} (\hbar\omega)^4 (\hat{a}^\dagger - \hat{a})^4$

$(\hat{a}^\dagger - \hat{a})^2 \,|\, n\rangle = \langle n|(\hat{a}^\dagger - \hat{a})(\hat{a}^\dagger - \hat{a}) \,|\, n\rangle = \langle n|(\hat{a}^\dagger - \hat{a})[\sqrt{n+1}\,|n+1\rangle - \sqrt{n}\,|n-1\rangle] =$

$[\sqrt{n+1}\,\sqrt{n+2}\,|n+2\rangle - n|n\rangle - \sqrt{n+1}\,\sqrt{n+1}\,|n\rangle + \sqrt{n}\,\sqrt{n-1}\,|n-2\rangle] =$

$[\sqrt{n+1}\,\sqrt{n+2}\,|n+2\rangle - (2n+1)|n\rangle + \sqrt{n}\,\sqrt{n-1}\,|n-2\rangle] =$

$[(\hat{a}^\dagger - \hat{a})^2 \,|\, n\rangle]^\dagger = \langle n|[(\hat{a}^\dagger - \hat{a})(\hat{a}^\dagger - \hat{a})]^\dagger = \langle n|[(\hat{a}^\dagger - \hat{a})^\dagger (\hat{a}^\dagger - \hat{a})^\dagger] = \langle n|[((\hat{a} - \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger)] = \langle n|(\hat{a}^\dagger - \hat{a})^2 =$

$[\langle n+2|\sqrt{n+1}\,\sqrt{n+2} - \langle n|(2n+1) + \langle n-2|\sqrt{n}\,\sqrt{n-1}\,]$ so

$\langle n|(\hat{a}^\dagger - \hat{a})^4 \,|\, n\rangle = \langle n|(\hat{a}^\dagger - \hat{a})^2 (\hat{a}^\dagger - \hat{a})^2 \,|\, n\rangle = [\langle n+2|\sqrt{n+1}\,\sqrt{n+2} - \langle n|(2n+1) + \langle n-2|\sqrt{n}\,\sqrt{n-1}\,]$ $[\sqrt{n+1}\,\sqrt{n+2}\,|n+2\rangle - (2n+1|n\rangle + \sqrt{n}\,\sqrt{n-1}\,|n-2\rangle]$

$= (n+1)(n+2) + (2n+1)^2 + n(n-1) = n^2 + 3n + 2 + 4n^2 + 4n + 1 + n^2 - n = 6n^2 + 6n + 3 = 3(2n^2 + 2n + 1)$

So finally

$\langle \hat{H}' \rangle = -\frac{m}{128 c^2} (\hbar\omega)^4 3(2n^2 + 2n + 1) = -\frac{3m(\hbar\omega)^4 (2n^2+2n+1)}{128 c^2}$

4) This is now a problem where we solve the one case in which we have a time dependent hamiltonian. It is the case of spin magnetic resonance. This is the basis for the NMR machine at the hospital when used with nuclei instead of

electrons. But we will use electrons to make things easy. I have heard this go my the name ESR. μSR where the electron is replaced by a muon is often used to study materials in solid state physics.

An electron is at rest in an oscillating magnetic field which points in the $\hat{z}$ direction with a magnitude $B = B_0 \cos(\omega t)$

a) construct the hamiltonian for the system given that $\vec{\mu}_e = \frac{-e}{mc}\hat{S}_e$

b) The electron starts out in a spin state in the $+\hat{x}$ direction. Calculate the spin state as a function of time, that is calculate $\begin{pmatrix} \chi_+(t) \\ \chi_-(t) \end{pmatrix}$. Be careful since the time translation operator in this case is $\hat{U}(t, t_0) = e^{-\left(\frac{i}{\hbar}\right)\int_{t_0}^{t} \hat{H}(t')\,dt'}$ (Hint: Be very careful using this. Remember that you want to break things up into eigenkets before you can get eigenvalues)

c) Find the probability of getting $\frac{+\hbar}{2}$ if you measure $\hat{S}_x$. Then use this to find the minimum B_0 required to force a complete spin flip in $\hat{S}_x$.

a) $\hat{H} = -\vec{\mu}\cdot\vec{B} = \frac{e}{mc}\hat{S}_z B_0 \cos(\omega t) = \frac{e}{mc} B_0 \cos(\omega t)\,\hat{S}_z = \gamma \cos(\omega t)\,\hat{S}_z$ where $\gamma = \frac{eB_0}{mc}$

b) The initial condition is: $\begin{pmatrix} \chi_+(t=0) \\ \chi_-(t=0) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

the time evolution operator is

$$\hat{U}(t, t_0) = e^{-\left(\frac{i}{\hbar}\right)\int_0^t \gamma\cos(\omega t')\hat{S}_z\,dt'} = e^{-\left(\frac{i}{\hbar}\right)\gamma\hat{S}_z\int_0^t \cos(\omega t')\,dt'} = e^{-\left(\frac{i}{\hbar}\right)\gamma\hat{S}_z\int_0^t \cos(\omega t')\,dt'} = e^{-\left(\frac{i}{\hbar}\right)\frac{\gamma}{\omega}\hat{S}_z \sin(\omega t)}$$

Now apply this to the initial condition to get the answer

$$\begin{pmatrix} \chi_+(t) \\ \chi_-(t) \end{pmatrix} = e^{-\left(\frac{i}{\hbar}\right)\frac{\gamma}{\omega}\hat{S}_z \sin(\omega t)}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-\left(\frac{i}{\hbar}\right)\frac{\gamma}{\omega}\hat{S}_z \sin(\omega t)}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$$

$$e^{-i\frac{\gamma}{2\omega}\sin(\omega t)}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{i\frac{\gamma}{2\omega}\sin(\omega t)}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\gamma}{2\omega}\sin(\omega t)} \\ e^{i\frac{\gamma}{2\omega}\sin(\omega t)} \end{pmatrix}$$

c) To do this we want to find P= $|\langle S_x + |\chi\rangle|^2$ the probability that our state is in the eigenstate of S_x with eigenvalue $\frac{+\hbar}{2}$

$$\langle S_x + |\chi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\gamma}{2\omega}\sin(\omega t)} \\ e^{i\frac{\gamma}{2\omega}\sin(\omega t)} \end{pmatrix} = \frac{1}{2}\left(e^{-i\frac{\gamma}{2\omega}\sin(\omega t)} + e^{i\frac{\gamma}{2\omega}\sin(\omega t)}\right) = \cos\left[\frac{\gamma}{2\omega}\sin(\omega t)\right]$$

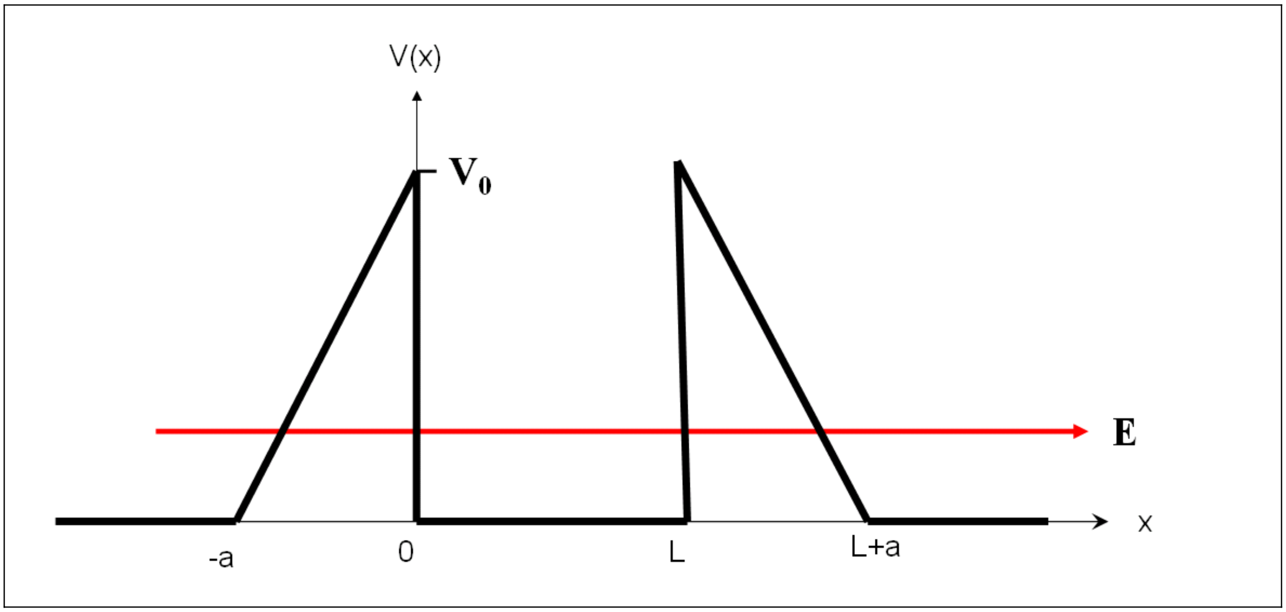
and $P = \cos^2\left[\frac{\gamma}{2\omega}\sin(\omega t)\right] = \cos^2\left[\frac{eB_0}{2mc\omega}\sin(\omega t)\right]$

For a flip we will need to make sure that $\frac{eB_0}{2mc\omega} \geqslant \pi$ so that as the $\sin(\omega t)$ term goes between 1 and -1, the probability can go between 0 and 1 so for the minimum B_0 we get $B_{0\min} = \frac{2\pi mc\omega}{e}$

5) For the potential below

a) calculate the transmission coefficient for the n=0 and n=1 states to penetrate through one of the barriers. Use for the energy, the energy of the n=1 state calculated using the WKB approximation.

b) Then assume $E = \hbar\omega$; L=10 fm (1 fm = 10^{-15} m); a=2 fm; mc^2=100 MeV; V_0=500 MeV and calculate the lifetime of the state. Remember $\hbar c$=200 MeV fm so we can calculate $\hbar\omega$. $E = \hbar\omega \rightarrow \omega = \frac{E}{\hbar} = \frac{Ec}{\hbar c}$



a) Finding the energy is trivial The WKB approx just gives the same thing it gives for the infinite potential well. i.e. V=0 and n=1

$\frac{1}{2\pi}\int_0^L \sqrt{2\,\text{mE}}\,dx = \left(n+\frac{1}{2}\right)\frac{\hbar}{2} \longrightarrow \quad \frac{1}{2\pi}\sqrt{2\,\text{mE}}\ \text{L} = \left(n+\frac{1}{2}\right)\frac{\hbar}{2} \quad \longrightarrow$

$E = \frac{\pi^2\hbar^2}{2\,\text{mL}^2}\left(n+\frac{1}{2}\right)^2 = \frac{\pi^2\hbar^2c^2}{2\,\text{mc}^2L^2}\frac{9}{4} = \frac{\pi^2(200\ \text{Mev-fm})^2}{2\,(100\ \text{MeV})\,(10\ \text{fm})^2}\frac{9}{4} = 44\ \text{MeV}$

The classical turning points are L and $L+a\left(1-\frac{E}{V_0}\right)$. The potential in this region is just
$V(x) = V_0 - V_0\frac{x-L}{a} = V_0\frac{(L+a)-x}{a}$

$\text{T}=\exp\left(-2\int_{x_1}^{x_2} dx\sqrt{\frac{2m}{\hbar^2}(V(x)-E)}\right)=\exp\left(-2\int_L^{L+a(1-E/V_0)} dx\sqrt{\frac{2m}{\hbar^2}\left(V_0(L+a)-\frac{x}{a}-E\right)}\right)=$

$\exp\left(-2\int_L^{L+a(1-E/V_0)} dx\sqrt{\frac{2m}{\hbar^2}\left[\frac{V_0(L+a)}{a}-E\right]-\frac{2\,\text{mV}_0}{\hbar^2 a}x}\right)=\exp\left(-2\int_L^{L+a(1-E/V_0)} dx\sqrt{b-\text{dx}}\right)=$

$\text{b}=\frac{2m}{\hbar^2}\left[\frac{V_0(L+a)}{a}-E\right] \quad d=\frac{2\,\text{mV}_0}{\hbar^2 a}$

$\text{T}=\exp\left(\frac{4(b-d\,x)^{3/2}}{3d}\right)$ {L, $L+a(1-E/V_0)$ in the exponential}$=\exp\left(\frac{4(b-d(L+a(1-E/V_0)))^{3/2}}{3d}-\frac{4(b-d\,L)^{3/2}}{3d}\right)$

$\text{b-dL}=\frac{2m}{\hbar^2}\left[\frac{V_0(L+a)}{a}-E\right]-\text{L}\frac{2\,\text{mV}_0}{\hbar^2 a}=\frac{2m}{\hbar^2}V_0\frac{L}{a}+\frac{2m}{\hbar^2}V_0-\frac{2m}{\hbar^2}E-\text{L}\frac{2\,\text{mV}_0}{\hbar^2 a}=\frac{2m}{\hbar^2}V_0-\frac{2m}{\hbar^2}E=\frac{2m}{\hbar^2}(V_0-E)$

$b-d\,L-\text{da}(1-E/V_0)=\text{b-d L-da+d}\frac{\text{aE}}{V_0}=\text{b-d L-}\frac{2\,\text{mV}_0}{\hbar^2 a}a+\frac{2\,\text{mV}_0}{\hbar^2 a}a\frac{E}{V_0}=\frac{2m}{\hbar^2}V_0-\frac{2m}{\hbar^2}E-\frac{2\,\text{mV}_0}{\hbar^2}+\frac{2\,\text{mE}}{\hbar^2}=0$

$\text{T}=\exp\left(-\frac{4(b-d\,L)^{3/2}}{3d}\right)=\exp\left(-\frac{4\left(\frac{2m}{\hbar^2}(V_0-E)\right)^{3/2}}{3\frac{2\,\text{mV}_0}{\hbar^2 a}}\right)=\exp\left(-\frac{4\left(\frac{2m}{\hbar^2}(V_0-E)\right)^{3/2}}{3\frac{2\,\text{mV}_0}{\hbar^2 a}}\right)$

$\frac{2m}{\hbar^2}=\frac{2\,\text{mc}^2}{\hbar^2c^2}=\frac{2\,(100\ \text{MeV})}{(200\ \text{MeV}-\text{fm})^2}=\frac{1}{200\ \text{MeV}-\text{fm}^2} \qquad \frac{2\,\text{mV}_0}{\hbar^2}=\frac{500\ \text{MeV}}{200\ \text{MeV}-\text{fm}^2}=2.5\ \text{fm}^{-2} \qquad \frac{2\,\text{me}}{\hbar^2}=\frac{44\ \text{MeV}}{200\ \text{MeV}-\text{fm}^2}=0.22\ \text{fm}^{-2}$

$\text{T}=\exp\left(-\frac{4\,(2.5-0.22)^{3/2}\ \text{fm}^{-3}}{3\left(\frac{2.5}{2}\right)\text{fm}^{-3}}\right)=\exp(-\ 3.67)=.0254$

Integrate[Sqrt[$b - d * x$], x]

$$-\frac{2\,(b-d\,x)^{3/2}}{3\,d}$$

b) P=fT=2$\frac{\omega}{2\pi}$T

$\tau = \frac{1}{P} = \frac{\pi}{T\omega}$ so $\tau c = \frac{\pi\hbar c}{T\hbar\omega} = \frac{\pi(200\text{ MeV}\cdot\text{fm})}{2.54\text{x}10^{-2}\ 44\text{ MeV}} = 562$ fm $\quad \tau = \frac{562\text{ fm}}{3\text{x}10^{8}\ m/s}\ \frac{1\ m}{10^{15}\text{ fm}} = 1.9\text{x}10^{-23}$s

Here are some constants :

$1\ \text{fm} = 10^{-15}\ m$

$\hbar\text{c} = 200\ \text{MeV}\cdot\text{fm}$

$\hbar = 1.05457148\times10^{-34}\ J\cdot s = 6.5821\ \times10^{-16}\ \text{eV}\cdot s$

$c = 3\times10^{8}\ m/s$

$1\ \text{eV} = 1.6\times10^{-19}\ J$

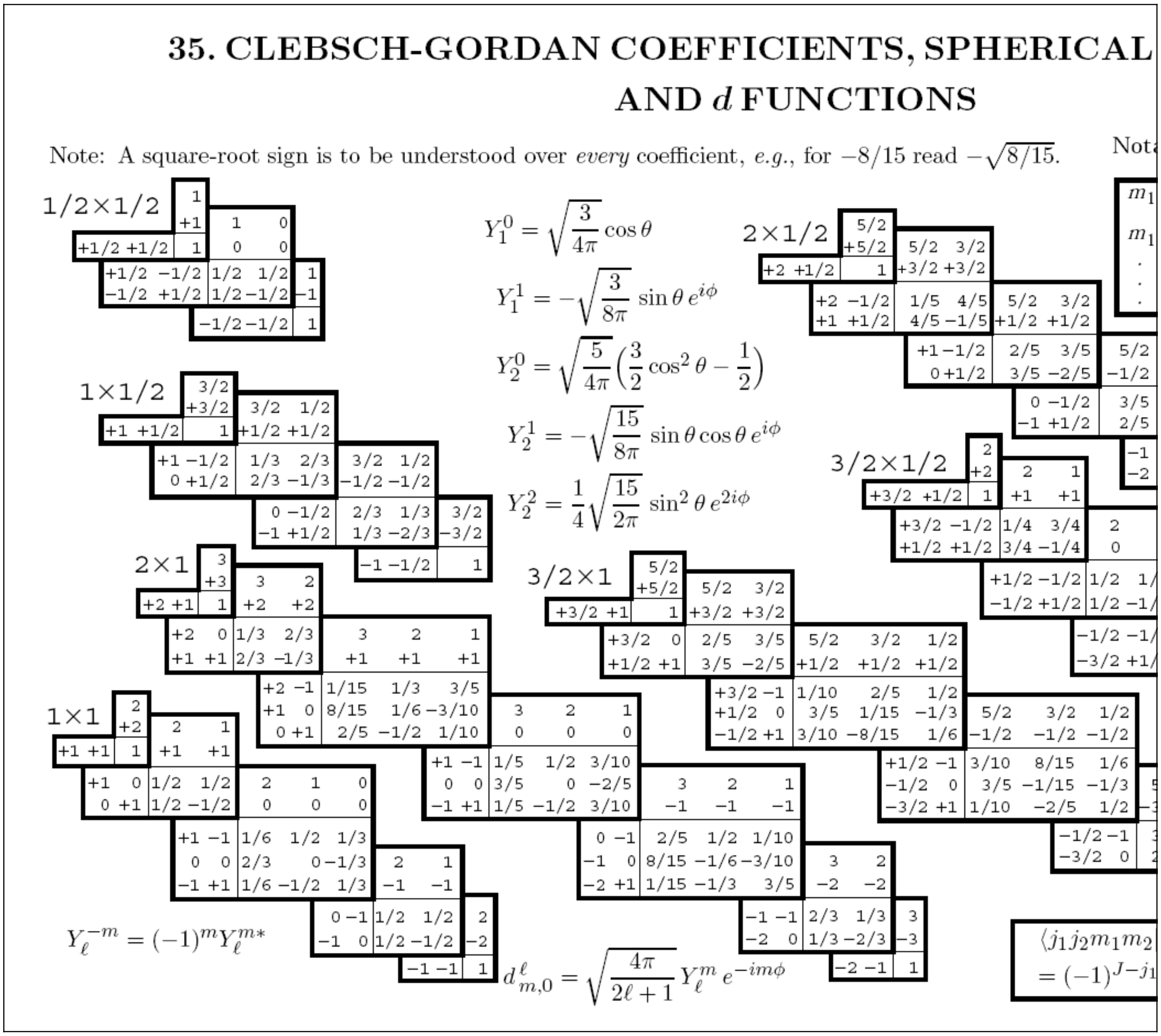
$$\mathcal{R} = \frac{\text{me}^4}{2\,\hbar^2} = 13.6\ \text{eV}$$

$$a_0 = \frac{\hbar^2}{\text{me}^2}$$

$$\alpha = \frac{e^2}{\hbar\text{c}} \sim \frac{1}{137}$$

Here is the taylor series.

$$f(x+h) = f(x) + \text{hf}'(x) + \frac{h^2}{2!}\ f''(x)\ ...$$



The following is missing in the CG tables $Y_0^{\,0} = \dfrac{1}{\sqrt{4\pi}}$

Reminders from first quarter

$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

eigenkets of $\hat{S}_z$ **are** $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **and** $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; **eigenkets of** $\hat{S}_x$ **are** $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$; **eigenkets of** $\hat{S}_y$ **are** $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$

$\hat{S}_+ \doteq \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\hat{S}_- \doteq \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Hydrogen atom

$\langle \text{nlsjm} \,|\, \hat{H}_0 \,| \text{nlsjm} \rangle = E_n = -\frac{Z^2 \mathcal{R}}{n^2}$

Here are the first set of wave function in spectroscopic notation where the notation 1s means n=1, l=0

remember: s: l=0 p: l=1 d: l=2 f: l=3 ...

1s $\varphi_{100} = \frac{2}{a_0{}^{3/2}} e^{-r/a_0} Y_0{}^0$

2s $\varphi_{200} = \frac{2}{(2 a_0)^{3/2}} \left(1 - \frac{r}{2 a_0}\right) e^{-r/a_0} Y_0{}^0$

2p $\varphi_{21\,m} = \frac{2}{\sqrt{3}\,(2 a_0)^{3/2}} \frac{r}{2 a_0} e^{-r/a_0} Y_1{}^m$

3s $\varphi_{300} = \frac{2}{3 (2 a_0)^{3/2}} \left[3 - 2 \frac{r}{a_0} + 2 \left(\frac{r}{3 a_0}\right)^2\right] e^{-r/a_0} Y_0{}^0$

3p $\varphi_{31\,m} = \frac{4\sqrt{2}}{9 (3 a_0)^{3/2}} \frac{r}{a_0} \left[1 - \frac{r}{6 a_0}\right] e^{-r/a_0} Y_1{}^m$

3d $\varphi_{32\,m} = \frac{2\sqrt{2}}{27\sqrt{5}\,(3 a_0)^{3/2}} \left(\frac{r}{a_0}\right)^2 e^{-r/a_0} Y_2{}^m$

Particle in an invite potential well

$$\boldsymbol{\varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad E_n = \frac{n^2 \hbar^2 \pi^2}{2\, mL^2}}$$ **, $n = 1,\ 2,\ \ldots$ for $0 < x < L$**

WKB:

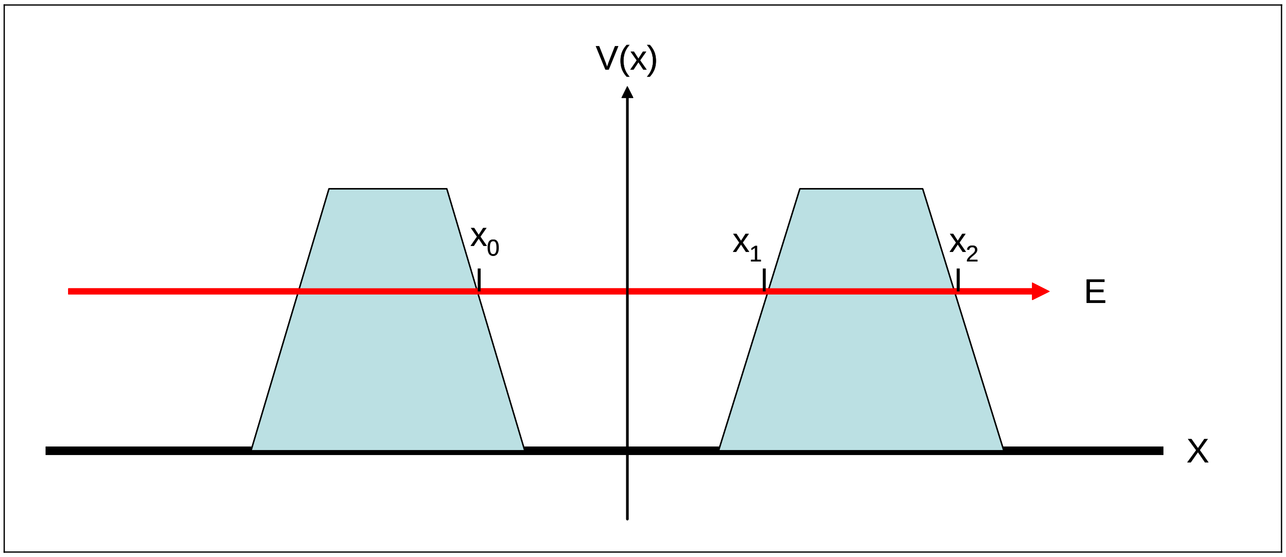
For a configuration as shown below the transmission probability in the WKB approximation is

T=exp(-2$\int_{x_1}^{x_2} \kappa \, dx$)   where $\kappa = \sqrt{\frac{2m}{\hbar^2} (V(x) - E)}$ so

T=exp$\left(-2 \int_{x_1}^{x_2} dx \sqrt{\frac{2m}{\hbar^2} (V(x) - E)}\right)$

where x_0, x_1, and x_2 are the classical turning points, so E= V(x), i.e. $V(x_0) = V(x_1) = V(x_2) = E$ .The energies allowed in the well will obey the condition

$\frac{1}{2\pi} \int_{x_0}^{x_1} \sqrt{2m(E - V(x))}\, dx = \left(n + \frac{1}{2}\right) \frac{\hbar}{2}$



Here are some integrals you may need.

1) $\int_0^{1/2} \sin(m\pi x)\sin(n\pi x)\, dx$

Integrate[Sin[m * Pi * x] * Sin[n * Pi * x], {x, 0, 1/2}]

$$\frac{n\cos\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)}{m^2\pi - n^2\pi} - \frac{m\cos\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)}{m^2\pi - n^2\pi}$$

2) $\int \sqrt{a - bx}\, dx$

Integrate[Sqrt[a - b * x], x]

$$-\frac{2(a - b x)^{3/2}}{3 b}$$