Combining Data from Different Measurements Using the BLUE Method

RHIC Winter School
Budapest
5-7 December 2007

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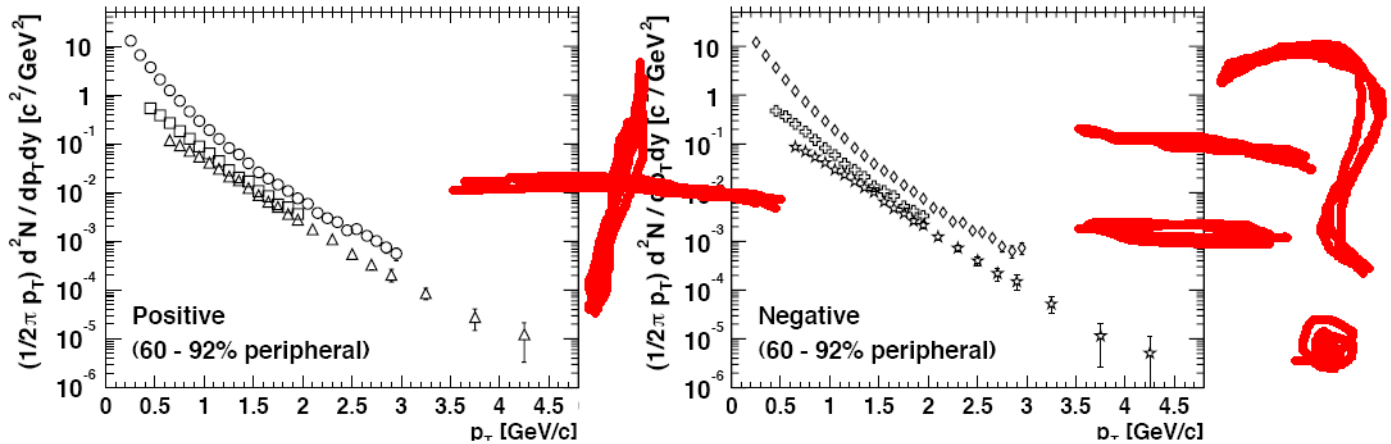
Topic

- Combining measurements with correlated uncertainties is a common need in high energy physics.



- We are going to have an overview of a simple way to do it

Outline

- Measurements and Errors
  - Common systematic uncertainties
- The BLUE Method
  - Method for combining correlated measurements
  - Finding the weights of the average
- Separation of Error types
  - Correlation coefficient
  - Handling different types of systematic errors
- Examples
  - D0 top quark mass, PHENIX illustration

Measurements and Errors

- Measurements of observable X: x_i with error σ

$$\sigma_i^2 = E\left(x_i - E\left(x_i\right)\right)^2$$

- Distribution is Poisson-like!
- Types of uncertainties
  - Statistical
  - Systematic (specific type: α)

$$\sigma_i^2 = \sigma_{stat,i}^2 + \sigma_{syst,i}^2$$

$$\sigma_{syst,i}^2 = \sum_\alpha \sigma_{syst\alpha,i}^2$$

- Independent measurements

$$\langle x \rangle = \frac{\sum_i \left(1/\sigma_i^2\right) x_i}{\sum_i \left(1/\sigma_i^2\right)}, \quad \frac{1}{\sigma^2} = \sum_i \frac{1}{\sigma_i^2}; \quad \chi^2 = \sum_i \frac{1}{\sigma_i^2}\left(\langle x \rangle - x_i\right)$$

Common Systematic Uncertainties

- Correlated measurements – covariance

$$C_{ij} = E\left(x_i - E\left(x_i\right)\right) E\left(x_j - E\left(x_j\right)\right) = \rho_{ij}\sigma_i\sigma_j \qquad C_{ii} = \sigma_i^2$$

- Separation of correlated errors

$$\left(\sigma^{corr}\right)^2 = \sigma_i^2 - \left(\sigma_i^{uncorr}\right)^2 = \sum_\alpha \left(\sigma_{syst\,\alpha,i}^{corr}\right)^2$$

$$\langle x \rangle = \frac{\sum_i \left(1/\left(\sigma_i^2 - \left(\sigma^{corr}\right)^2\right)\right) x_i}{\sum_i \left(1/\left(\sigma_i^2 - \left(\sigma^{corr}\right)^2\right)\right)}, \qquad \sigma^2 = \sum_i \frac{1}{\left(1/\left(\sigma_i^2 - \left(\sigma^{corr}\right)^2\right)\right)} + \left(\sigma^{corr}\right)^2$$

- Two measurements

$$\langle x \rangle = \frac{x_1\left(\sigma_2^{uncorr}\right)^2 + x_2\left(\sigma_1^{uncorr}\right)^2}{\left(\sigma_1^{uncorr}\right)^2 + \left(\sigma_2^{uncorr}\right)^2}, \qquad \sigma^2 = \frac{\sigma_1^2\sigma_2^2 - \left(\sigma^{corr}\right)^4}{\sigma_1^2 + \sigma_2^2 - 2\left(\sigma^{corr}\right)^2}$$

Best Linear Unbiased Estimation

- A linear combination of correlated measurements x_i of the same observable X

$$\langle x \rangle = \sum_i w_i x_i$$

- Unbiased, ie. E(X) = E(<X>) ↔ norm condition

$$\sum_i w_i = 1$$

- Let the possible variance be minimal

$$\sigma^2 = w^T \mathbf{C} \mathbf{w} = \sum_{i,j} C_{ij} w_i w_i \qquad C_{ij} = \rho_{ij} \sigma_i \sigma_j$$

Ways to Find BLUE Weights (1)

- Looking for the N values of w_i that minimize σ
- 3 equivalent ways

1. Elimination
   - Use norm constraint to get rid of (say) w_1
   - Minimize σ for $w_2..w_N$ regarded as independent

1. Make use of Lagrangian multipliers

Ways to Find BLUE Weights (2)

3. Weighted sum of squares
   - Define a generalized χ^2

$$\chi^2 = \Delta \mathbf{x}^T C^{-1} \Delta \mathbf{x} = \sum_{i,j} w_{ij} \Delta x_i \Delta x_j$$

$$w_{ij} = C_{ij}^{-1} \quad \Delta x_i = x_i - \langle x_i \rangle$$

   - Minimize it and express <x> and σ

$$\langle x \rangle = \frac{\sum_{i,j} w_{ij} x_i}{\sum_{i,j} w_{ij}} \quad \sigma^2 = \frac{1}{\sum_{i,j} w_{ij}}$$

BLUE for Two Measurements

- Weighted average of the measurements

$$\langle x \rangle = w_1 x_1 + w_2 x_2 \quad w_1 + w_2 = 1$$

$$C^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2 (1-\rho^2)} \begin{bmatrix} \sigma_1^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$\chi^2 = \mathbf{\Delta x}^T C^{-1} \mathbf{\Delta x} = \frac{(\Delta x_1)^2 \sigma_2^2 + (\Delta x_2)^2 \sigma_1^2 - 2\Delta x_1 \Delta x_2 \rho\sigma_1\sigma_2}{\sigma_1^2 \sigma_2^2 (1-\rho^2)}$$

- From here the weights

$$w_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \qquad w_2 = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

- And errors

$$\sigma^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 - 2 w_1 w_2 \rho\sigma_1\sigma_2$$

$$\sigma_{stat}^2 = w_1^2 \sigma_{1\,stat}^2 + w_2^2 \sigma_{2\,stat}^2 \qquad \sigma_{syst}^2 = \sigma^2 - \sigma_{stat}^2$$

Special Cases

- $\rho=0$ (uncorrelated)
  - Same as the pure statistical method
- $\rho=\sigma_1/\sigma_2$
  - $w_2=0$. The weaker measurement is ignored
- $\rho>\sigma_1/\sigma_2$ (overcorrelated)
  - $w_2<0$ (!) . The two measurements move together so strongly that they are on the same side of the “average”
- $\rho=1$ (completely correlated)
  - $\sigma=0$ (!)

Correlation Coefficient

- Define α-type correlation coefficitens

$$C_{ij} = \delta_{ij}\sigma^2_{stat,i} + \sum_\alpha \rho_{syst\alpha,ij}\sigma_{syst\alpha,i}\sigma_{syst\alpha,j}$$

- More specifically

$$\left(\sigma^{corr}\right)^2 = C_{12} = \sum_\alpha \rho_{syst\alpha}\sigma_{syst\alpha,1}\sigma_{syst\alpha,2}$$

$$\rho = \frac{\sum_\alpha \rho_{syst\alpha}\sigma_{syst\alpha,1}\sigma_{syst\alpha,2}}{\sigma_1\sigma_2}$$

- Fully correlated case

$$\left(\sigma^{corr}\right)^2 = \sum_{corr.syst.\alpha}\sigma_{syst\alpha,1}\sigma_{syst\alpha,2}$$

$$\rho = \frac{\sum_{corr.syst.\alpha}\sigma_{syst\alpha,1}\sigma_{syst\alpha,2}}{\sigma_1\sigma_2}$$

Separation of Error Types

For two measurements

- General case

$$\sigma^2_{syst\alpha} = w_1^2\sigma^2_{syst\alpha,1} + w_2^2\sigma^2_{syst\alpha,2} + 2\rho_{syst\alpha}\sigma_{syst\alpha,1}\sigma_{syst\alpha,2}$$

- Fully correlated errors

$$\sigma^2_{syst\alpha} = w_1\sigma^2_{syst\alpha,1} + w_2\sigma^2_{syst\alpha,2} \qquad \rho_{syst\alpha} = 1$$

Note:

Fully correlated errors scale linearly with weight while uncorrelated errors quadratically

Warning:

Watch out when assuming full correlation

A Numerical Example

D0 Experiment (Fermilab) top quark mass meas.

1. Run-I dilepton channel

$$m_t = 168.4 \pm 12.3(\text{stat}) \pm 13.6(\text{syst})\,\text{GeV/c}^2 = 168.4 \pm 12.8\,\text{GeV/c}^2$$

2. Run-II lepton-jet channel

$$m_t = 180.1 \pm 3.6(\text{stat}) \pm 3.9(\text{syst})\,\text{GeV/c}^2 = 180.1 \pm 5.3\,\text{GeV/c}^2$$

3. So that the combined mass

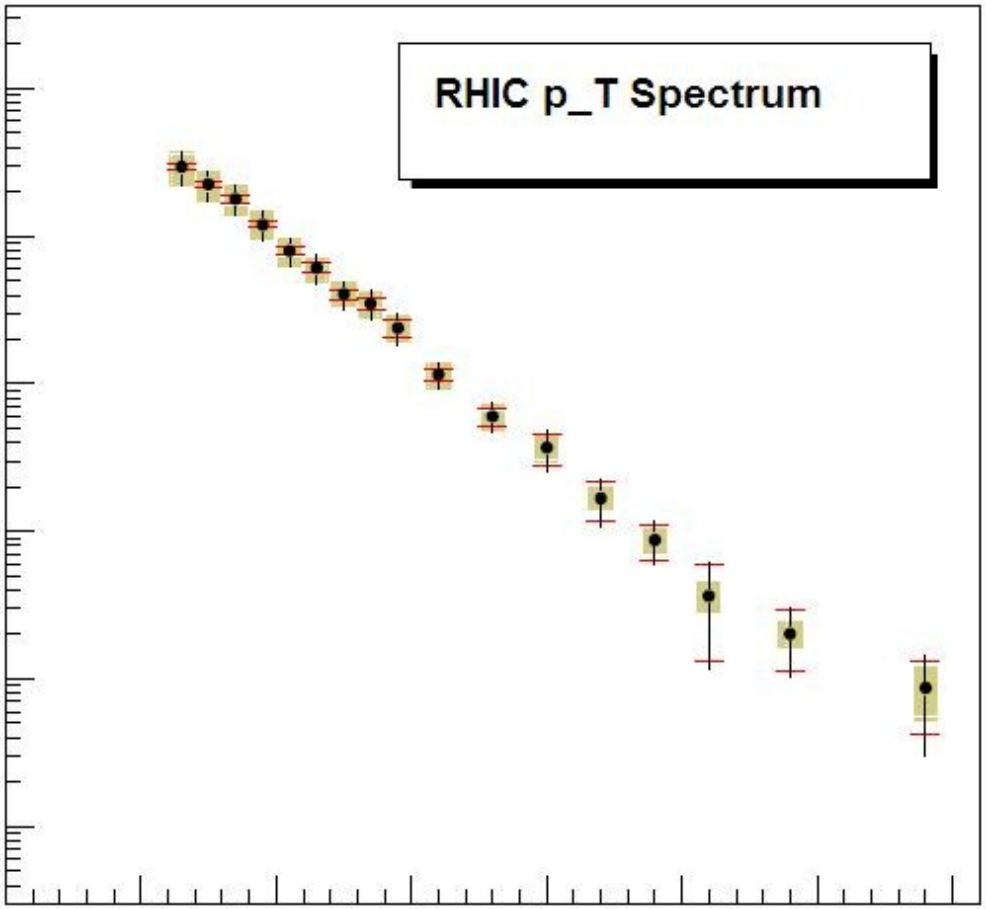
$$\rho = 0.17 \quad \Leftrightarrow \quad \sigma^{corr} = \sqrt{0.17 \cdot 12.8 \cdot 5.3}\,\text{GeV/c}^2 = 3.4\,\text{GeV/c}^2$$

$$m_t = 179.0 \pm 3.5(\text{stat}) \pm 3.8(\text{syst})\,\text{GeV/c}$$

$$\chi^2 = 0.81$$

Example: RHIC P_T Spectrum

Yield



p_T

Example: RHIC Spectrum Errors

Uncertainty	Error type	Correlation between runs
Statistical	A	0
Fidutial & acceptancy cuts	C	0
Energy loss cut	C	0
Matching cut	C	0
Run-by-run variation	C	0
PID cut	B	0
p-p feed down correction	B	100%
BBC luminosity	C	100%

- Type **A**: Point-to-point errors
- Type **B**: p_T-correlated errors
- Type **C**: Absolute normalization errors

Partly correlated

Illustration: $\sigma_{type\ \alpha}/\sigma$ Run2 Spectrum

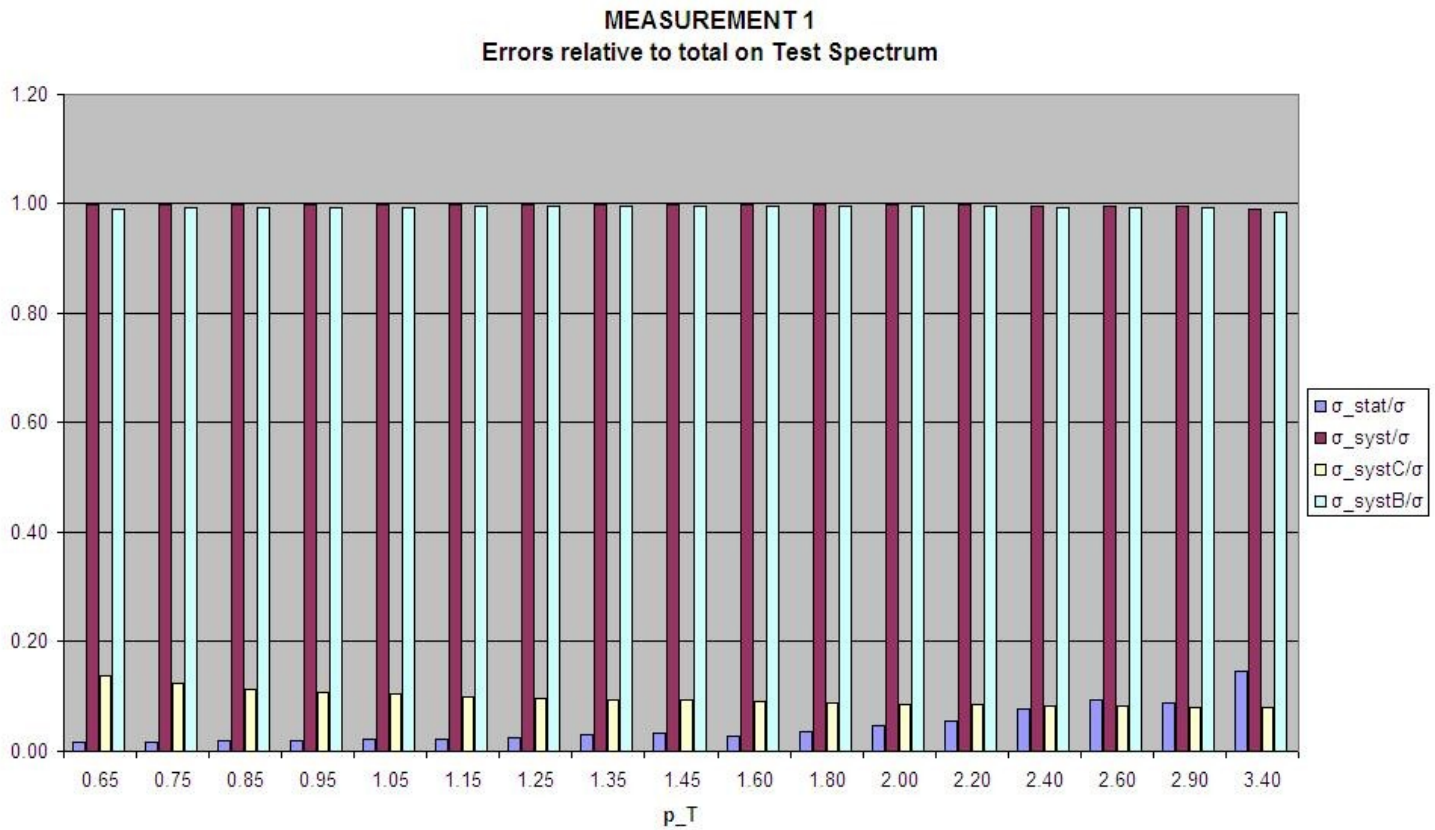


Illustration: $\sigma_{type\ \alpha}/\sigma$ Run3 Spectrum

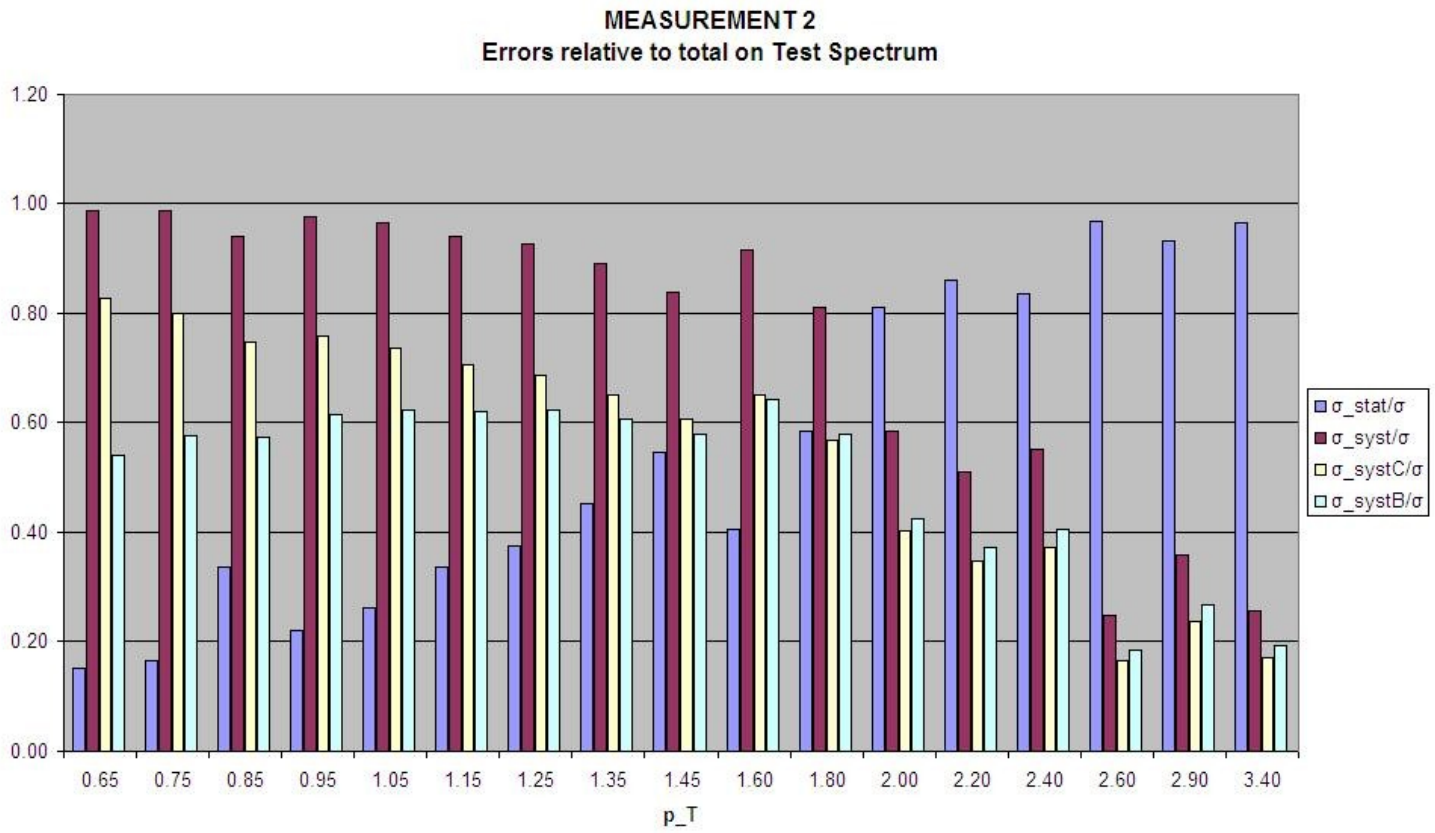
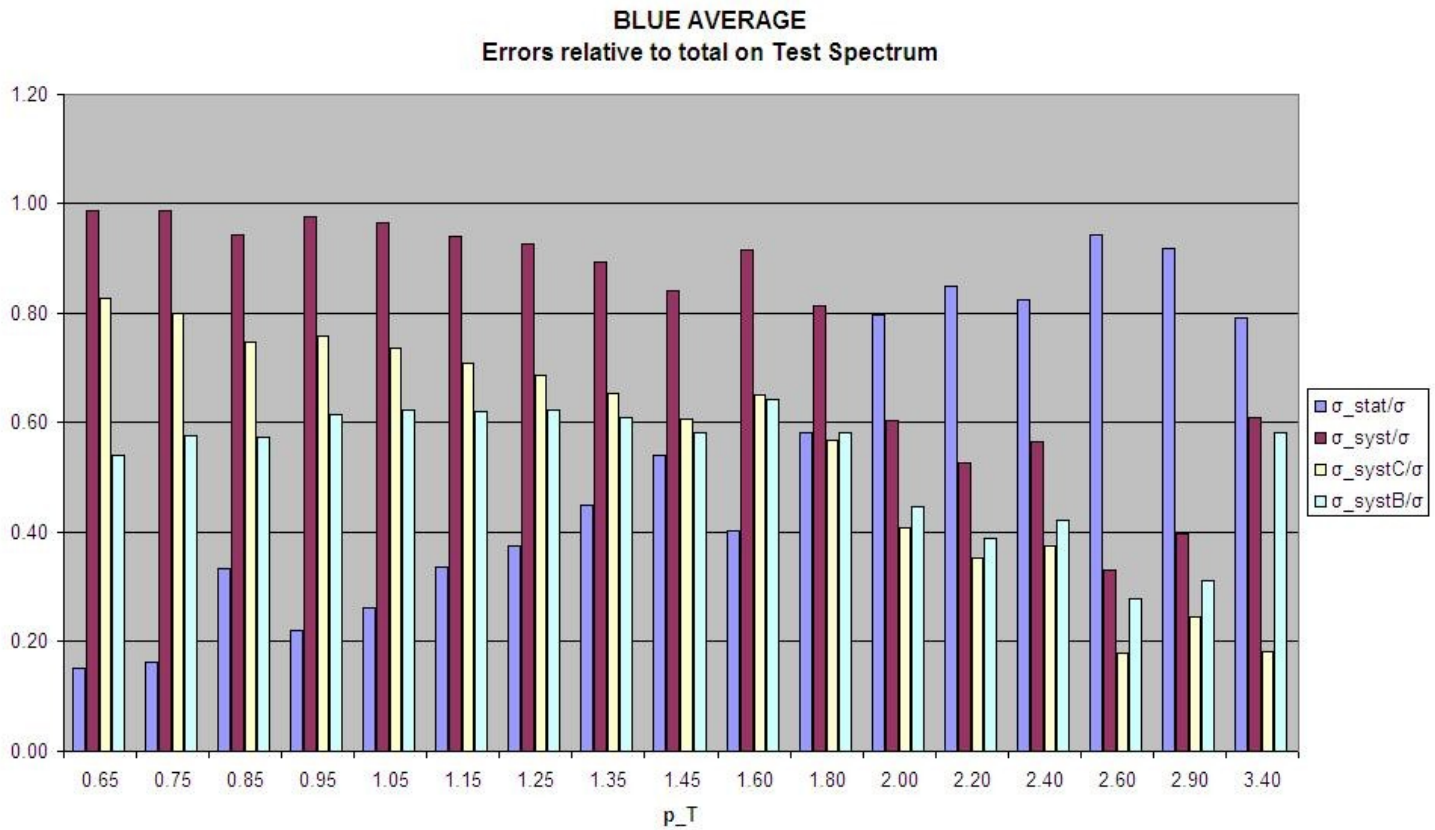


Illustration: $\sigma_{type\ \alpha}/\sigma$ BLUE Spectrum



Summary

- Combining measurements with correlated uncertainties is a common need in high energy physics
- We had an overview on the BLUE method for this purpose
- Decomposition of the total errors into the usual components was shown
- We have seen some examples of different usages

Credits

- L. Lyons, D. Gibaut, and P. Clifford
  Nucl. Inst. Meth. A.270, 110 (1988)
- Andrea Valassi (CERN):
  „Combining correlated measurements of several different physical quantities”
  Nucl.Inst.Meth.A500, 391-405 (2003)
- David d’Enterria
  PHENIX analyses using BLUE, note
- T. Csörgő & J. Sziklai
  Useful advises on combining Phenix spectra
- The Fulbright Commission