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Spherical Coordinates

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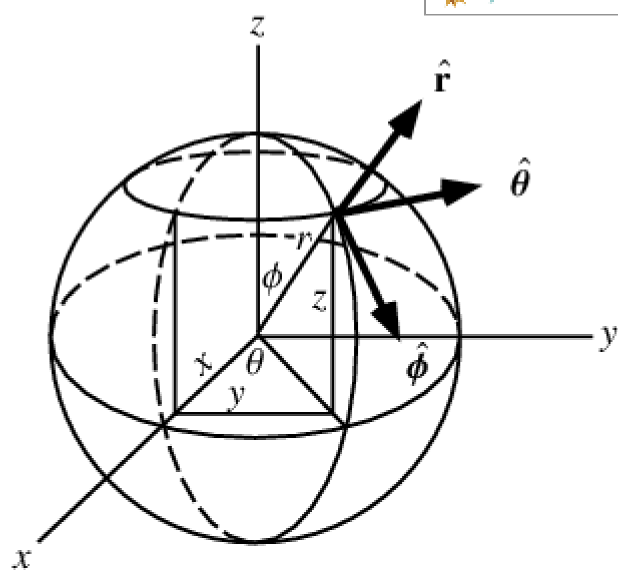

Spherical coordinates are a system of curvilinear coordinates that are natural fo
positions on a sphere or spheroid. Define θ to be the azimuthal angle in the *xy*
the *x*-axis with $0 \le \theta < 2\pi$ (denoted λ when referred to as the longitude), ϕ
polar angle from the *z*-axis with $0 \le \phi \le \pi$ (colatitude, equal to $\phi = 90° - \delta$
the latitude), and *r* to be distance (radius) from a point to the origin.

Unfortunately, the convention in which the symbols θ and ϕ are reversed is fr
used, especially in physics, leading to unnecessary confusion. The symbol ρ is
also used in place of *r*. Arfken (1985) uses (r, ϕ, θ), whereas Beyer (1987) use
Be very careful when consulting the literature.

In this work, the symbols for the azimuth, zenith, and radial coordinates are tak
and *r*, respectively (Zwillinger 1985, pp. 297-298). Note that this definition pro
extension of the usual polar coordinates notation, with θ remaining the angle i
and ϕ becoming the angle out of the plane.

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\phi = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{r}\right) = \cos^{-1}\left(\frac{z}{r}\right),$$

where $r \in [0, \infty)$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$. In terms of Cartesian coordir

$$x = r\cos\theta\sin\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\phi.$$

The scale factors are

$$h_r = 1$$

$$h_\theta = r\sin\phi$$

$$h_\phi = r,$$

so the metric coefficients are

$$g_{rr} = 1$$

$$g_{\theta\theta} = r^2\sin^2\phi$$

$$g_{\phi\phi} = r^2.$$

The line element is

$$d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + r\sin\phi\,d\theta\,\hat{\boldsymbol{\theta}},$$

the area element

$$d\mathbf{a} = r^2\sin\phi\,d\theta\,d\phi\,\hat{\mathbf{r}},$$

and the volume element

$$dV = r^2\sin\phi\,d\phi\,d\theta\,dr.$$

The Jacobian is

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| = r^2\left|\sin\phi\right|.$$

The position vector is

$$\mathbf{r} \equiv \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix},$$

so the unit vectors are

$$\hat{\mathbf{r}} \equiv \frac{\frac{d\mathbf{r}}{dr}}{\left|\frac{d\mathbf{r}}{dr}\right|} = \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}$$

$$\hat{\boldsymbol{\theta}} \equiv \frac{\frac{d\mathbf{r}}{d\theta}}{\left|\frac{d\mathbf{r}}{d\theta}\right|} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

$$\hat{\boldsymbol{\phi}} \equiv \frac{\frac{d\mathbf{r}}{d\phi}}{\left|\frac{d\mathbf{r}}{d\phi}\right|} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix}.$$

Derivatives of the unit vectors are

$$\frac{\partial\hat{\mathbf{r}}}{\partial r} = \mathbf{0}$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial r} = \mathbf{0}$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial r} = \mathbf{0}$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \begin{bmatrix} -\sin\theta\sin\phi \\ \cos\theta\sin\phi \\ 0 \end{bmatrix} = \sin\phi\,\hat{\boldsymbol{\theta}}$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = \begin{bmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} = -\cos\phi\,\hat{\boldsymbol{\phi}} - \sin\phi\,\hat{\mathbf{r}}$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\theta} = \begin{bmatrix} -\sin\theta\cos\phi \\ \cos\theta\cos\phi \\ 0 \end{bmatrix} = \cos\phi\,\hat{\boldsymbol{\theta}}$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\phi} = \begin{bmatrix} \cos\theta \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix} = \hat{\boldsymbol{\phi}}$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} = \mathbf{0}$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} =$$

$$\begin{bmatrix} -\cos\theta\sin\phi \\ -\sin\theta\sin\phi \\ -\cos\phi \end{bmatrix} = -\hat{\mathbf{r}}.$$

The gradient is

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\phi} + \frac{1}{r\sin\phi}\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta},$$

so

$$\begin{aligned}
\nabla_r\hat{\mathbf{r}} &= \mathbf{0} \\
\nabla_r\hat{\boldsymbol{\theta}} &= \mathbf{0} \\
\nabla_r\hat{\boldsymbol{\phi}} &= \hat{\mathbf{0}} \\
\nabla_\theta\hat{\mathbf{r}} &= \frac{\sin\phi\,\hat{\boldsymbol{\theta}}}{r\sin\phi} = \frac{1}{r}\hat{\boldsymbol{\theta}} \\
\nabla_\theta\hat{\boldsymbol{\theta}} &= -\frac{\cos\phi\,\hat{\boldsymbol{\phi}} + \sin\phi\,\hat{\mathbf{r}}}{r\sin\phi} = -\frac{\cot\phi}{r}\hat{\boldsymbol{\phi}} - \frac{1}{r}\hat{\mathbf{r}} \\
\nabla_\theta\hat{\boldsymbol{\phi}} &= \frac{\cos\phi\,\hat{\boldsymbol{\phi}}}{r\sin\phi} = \frac{1}{r}\cot\phi\,\hat{\boldsymbol{\theta}}.
\end{aligned}$$

Now, since the connection coefficients are given by $\Gamma^i_{jk} = \hat{\mathbf{x}}_i \cdot (\nabla_k \hat{\mathbf{x}}_j)$,

$$\begin{aligned}
\Gamma^\theta &= \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\cot\phi}{r} & 0 \end{bmatrix} \\
\Gamma^\phi &= \begin{bmatrix} 0 & 0 & \frac{1}{r} \\ 0 & -\frac{\cot\phi}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\Gamma^r &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{r} & 0 \\ 0 & 0 & -\frac{1}{r} \end{bmatrix}.
\end{aligned}$$

The divergence is

$$\nabla \cdot \mathbf{F} = A^k_{,k} + \Gamma^k_{jk} A^j$$

$$= [A^r_{,r} + (\Gamma^r_{rr} A^r + \Gamma^r_{\theta r} A^\theta + \Gamma^r_{\phi r} A^\phi]$$

$$+[A^{\theta}_{,\theta} + (\Gamma^{\theta}_{r\theta}A^r + \Gamma^{\theta}_{\theta\theta}A^{\theta} + \Gamma^{\theta}_{\phi\theta}A^{\phi})]$$

$$+[A^{\phi}_{,\phi} + (\Gamma^{\phi}_{r\phi}A^r + \Gamma^{\phi}_{\theta\phi}A^{\theta} + \Gamma^{\phi}_{\phi\phi}A^{\phi})]$$

$$= \tfrac{1}{g_r}\tfrac{\partial A^r}{\partial r} + \tfrac{1}{g_\theta}\tfrac{\partial A^\theta}{\partial \theta} + \tfrac{1}{g_\phi}\tfrac{\partial A^\phi}{\partial \phi} + (0+0+0)$$

$$+ \left(\tfrac{1}{r}A^r + 0 + \tfrac{\cot\phi}{r}A^\phi\right) + \left(\tfrac{1}{r}A^r + 0 + 0\right)$$

$$= \tfrac{\partial}{\partial r}A^r + \tfrac{2}{r}A^r + \tfrac{1}{r\sin\phi}\tfrac{\partial}{\partial\theta}A^\theta + \tfrac{1}{r}\tfrac{\partial}{\partial\phi}A^\phi + \tfrac{\cot\phi}{r}A^\phi,$$

or, in vector notation,

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \left(\frac{2}{r} + \frac{\partial}{\partial r}\right)F_r + \left(\frac{1}{r}\frac{\partial}{\partial\phi} + \frac{\cot\phi}{r}\right)F_\phi + \frac{1}{\sin\phi}\frac{\partial F_\theta}{\partial\theta}\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi F_\phi) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial\theta}.\end{aligned}$$

The covariant derivatives are given by

$$A_{j;k} = \frac{1}{g_{kk}}\frac{\partial A_j}{\partial x_k} - \Gamma^i_{jk}A_i,$$

so

$$\begin{aligned}A_{r;r} &= \frac{\partial A_r}{\partial r} - \Gamma^i_{rr}A_i = \frac{\partial A_r}{\partial r}\\ A_{r;\theta} &= \frac{1}{r\sin\phi}\frac{\partial A_r}{\partial\theta} - \Gamma^i_{r\theta} = \frac{1}{r\sin\phi}\frac{\partial A_r}{\partial\theta} - \Gamma_{r\theta}A_\theta\\ &= \frac{1}{r\sin\phi}\frac{\partial A_r}{\partial\phi} - \frac{A_\theta}{r}\\ A_{r;\phi} &= \frac{1}{r}\frac{\partial A_r}{\partial\phi} - \Gamma^i_{r\phi}A_i = \frac{1}{r}\frac{\partial A_r}{\partial\phi} - \Gamma^\phi_{r\phi}A_\phi\\ &= \frac{1}{r}\left(\frac{\partial A_r}{\partial\phi} - A_\phi\right)\\ A_{\theta;r} &= \frac{\partial A_\theta}{\partial r} - \Gamma^i_{\theta r}A_i = \frac{\partial A_\theta}{\partial r}\\ A_{\theta;\theta} &= \frac{1}{r\sin\phi}\frac{\partial A_\phi}{\partial\theta} - \Gamma^i_{\theta\theta}A_i\\ &= \end{aligned}$$

$$\begin{aligned} & \frac{1}{r\sin\phi}\partial A_\theta\partial\theta - \Gamma^\phi_{\theta\theta}A_\phi - \Gamma^r_{\theta\theta}A_r \\ &= \frac{1}{r\sin\phi}\frac{\partial A_\theta}{\partial\theta} + \frac{\cot\phi}{r}A_\phi + \frac{A_r}{r} \\ A_{\theta;\phi} &= \frac{1}{r}\frac{\partial A_\theta}{\partial r} - \Gamma^i_{\phi r}A_i\frac{\partial A_\theta}{\partial\phi} \\ A_{\phi;r} &= \frac{\partial A_\phi}{\partial r} - \Gamma^i_{\phi r}A_i = \frac{\partial A_\phi}{r} \\ A_{\phi;\theta} &= \frac{1}{r\sin\phi}\frac{\partial A_\phi}{\partial\theta} - \Gamma^i_{\phi\theta}A_i = \frac{1}{r\sin\phi}\frac{\partial A_\phi}{\partial\theta} - \Gamma^\theta_{\phi\theta} \\ &= \frac{1}{r\sin\phi}\frac{\partial A_\phi}{\partial\theta} - \frac{\cot\phi}{r}A_\theta \\ A_{\phi;\phi} &= \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} - \Gamma^i_{\phi\phi}A_i = \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} - \Gamma^r_{\phi\phi}A_r \\ &= \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{A_r}{r}. \end{aligned}$$

The commutation coefficients are given by

$$c^\mu_{\alpha\beta}\vec{e}_\mu = [\vec{e}_\alpha, \vec{e}_\beta] = \nabla_\alpha\vec{e}_\beta - \nabla_\beta\vec{e}_\alpha$$

$$[\hat{\mathbf{r}}, \hat{\mathbf{r}}] = [\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}] = [\hat{\phi}, \hat{\phi}] = \mathbf{0},$$

so $c^\alpha_{rr} = c^\alpha_{\theta\theta} = c^\alpha_{\phi\phi} = 0$, where $\alpha = r, \theta, \phi$.

$$[\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}] = -[\hat{\boldsymbol{\theta}}, \hat{\mathbf{r}}] = \nabla_r\hat{\boldsymbol{\theta}} - \nabla_\theta\hat{\mathbf{r}} = \mathbf{0} - \frac{1}{r}\hat{\boldsymbol{\theta}} = -\frac{1}{r}\hat{\boldsymbol{\theta}},$$

so $c^\theta_{r\theta} = -c^\theta_{\theta r} = -\frac{1}{r}$, $c^r_{r\theta} = c^\phi_{r\theta} = 0$.

$$[\hat{\mathbf{r}}, \hat{\phi}] = -[\hat{\phi}, \hat{\mathbf{r}}] = \mathbf{0} - \frac{1}{r}\hat{\phi} = -\frac{1}{r}\hat{\phi},$$

so $c^\phi_{r\phi} = -c^\phi_{\phi r} = \frac{1}{r}$.

$$[\hat{\boldsymbol{\theta}}, \hat{\phi}] = -[\hat{\phi}, \hat{\boldsymbol{\theta}}] = \frac{1}{r}\cot\phi\hat{\boldsymbol{\theta}} - \mathbf{0} = \frac{1}{r}\cot\phi\hat{\boldsymbol{\theta}},$$

so

$$c^{\theta}_{\theta\phi} = -c^{\theta}_{\phi\theta} = \frac{1}{r}\cot\phi.$$

Summarizing,

$$\begin{aligned} c^r &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ c^{\theta} &= \begin{bmatrix} 0 & -\frac{1}{r} & 0 \\ \frac{1}{r} & 0 & \frac{1}{r}\cot\phi \\ 0 & -\frac{1}{r}\cot\phi & 0 \end{bmatrix} \\ c^{\phi} &= \begin{bmatrix} 0 & 0 & -\frac{1}{r} \\ 0 & 0 & 0 \\ \frac{1}{r} & 0 & 0 \end{bmatrix}. \end{aligned}$$

Time derivatives of the position vector are

$$\begin{aligned} \dot{\mathbf{r}} &= \begin{bmatrix} \cos\theta\sin\phi\,\dot{r} - r\sin\theta\sin\phi\,\dot{\theta} + r\cos\theta\cos\phi\,\dot{\phi} \\ \sin\theta\sin\phi\,\dot{r} + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi} \\ \cos\phi\,\dot{r} - r\sin\phi\,\dot{\phi} \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}\dot{r} + r\sin\phi\begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}\dot{\theta} + r\begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ -\sin\phi \end{bmatrix}\dot{\phi} \\ &= \dot{r}\,\hat{\mathbf{r}} + r\sin\phi\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\phi}\,\hat{\boldsymbol{\phi}}. \end{aligned}$$

The speed is therefore given by

$$v \equiv |\dot{\mathbf{r}}| = \sqrt{\dot{r}^2 + r^2\sin^2\phi\dot{\theta}^2 + r^2\dot{\phi}^2}.$$

The acceleration is

$$\begin{aligned} \ddot{x} &= (-\sin\theta\sin\phi\dot{\theta}\dot{r} + \cos\theta\cos\phi\dot{r}\dot{\phi} + \cos\theta\sin\phi\ddot{r}) \\ &\quad -(\sin\theta\sin\phi\dot{r}\dot{\theta} + r\cos\theta\sin\phi\dot{\theta}^2 + r\sin\theta\cos\phi\dot{\theta}\dot{\phi} \\ &\quad +r\sin\theta\sin\phi\ddot{\theta}) + (\cos\theta\cos\phi\dot{r}\dot{\phi} - r\sin\theta\cos\dot{\theta}\dot{\phi} \\ &\quad -r\cos\theta\sin\phi\dot{\phi}^2 + r\cos\theta\cos\phi\ddot{\phi}) \\ &= -2\sin\theta\sin\phi\dot{\theta}\dot{r} + 2\cos\theta\cos\phi\dot{r}\dot{\phi} - 2r\sin\theta\cos\phi\dot{\theta}\dot{\phi} \\ &\quad +\cos\theta\sin\phi\ddot{r} - r\sin\theta\sin\phi\ddot{\theta} + r\cos\theta\cos\phi\ddot{\phi} \end{aligned}$$

$$
\begin{aligned}
& -r\cos\theta\sin\phi(\dot\theta^2+\dot\phi^2) \\
\ddot y &= (\sin\theta\sin\phi\ddot r + r\cos\theta\sin\phi\dot\theta + r\cos\phi\sin\theta\dot\phi) \\
& +(\cos\theta\sin\phi\dot r\dot\theta - r\sin\theta\sin\phi\dot\theta^2 + r\cos\theta\cos\phi\dot\theta\dot\phi \\
& +r\cos\theta\sin\phi\ddot\theta) + (\sin\theta\cos\phi\dot r\dot\phi + r\cos\theta\cos\phi\dot\theta\dot\phi \\
& -r\sin\theta\sin\phi\dot\phi^2 + r\sin\theta\cos\phi\ddot\phi) \\
&= 2\cos\theta\sin\phi\dot\theta\dot r + 2\sin\theta\cos\phi\dot r\dot\phi + 2r\cos\theta\cos\phi\dot\theta\dot\phi \\
& +\sin\theta\sin\phi\ddot r + r\cos\theta\sin\phi\ddot\theta + r\sin\theta\cos\phi\ddot\phi \\
& -r\sin\theta\sin\phi(\dot\theta^2+\dot\phi^2) \\
\ddot z &= (\cos\phi\ddot r - \sin\phi\dot r\dot\phi) - (\dot r\sin\phi\dot\phi + r\cos\phi\dot\phi^2 + r\sin\phi\ddot\phi) \\
&= -r\cos\phi\dot\phi^2 + \cos\phi\ddot r - 2\sin\phi\dot\phi\dot r - r\sin\phi\ddot\phi.
\end{aligned}
$$

Plugging these in gives

$$
\begin{aligned}
\ddot{\mathbf r} &= (\ddot r - r\dot\phi^2)\begin{bmatrix}\cos\theta\sin\phi\\ \sin\theta\sin\phi\\ \cos\phi\end{bmatrix} \\
& +(2r\cos\phi\dot\theta\dot\phi + r\sin\phi\ddot\theta)\begin{bmatrix}-\sin\theta\\ \cos\theta\\ 0\end{bmatrix} \\
& +(2\dot r\dot\phi + r\ddot\phi)\begin{bmatrix}\cos\theta\cos\phi\\ \sin\theta\cos\phi\\ -\sin\phi\end{bmatrix} - r\sin\phi\dot\theta^2\begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix},
\end{aligned}
$$

but

$$
\begin{aligned}
\sin\phi\hat{\mathbf r} + \cos\phi\hat{\boldsymbol\phi} &= \begin{bmatrix}\cos\theta\sin^2\phi + \cos\theta\cos^2\phi\\ \sin\theta\sin^2\phi + \sin\theta\cos^2\phi\\ 0\end{bmatrix} \\
&= \begin{bmatrix}\cos\theta\\ \sin\theta\\ 0\end{bmatrix},
\end{aligned}
$$

so

$$\ddot{\mathbf r} =$$

$$
\begin{aligned}
& (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{r}} + (2r\cos\phi\dot{\theta}\dot{\phi} + 2\sin\phi\dot{\theta}\dot{r} + r\sin\phi\ddot{\theta})\hat{\boldsymbol{\theta}} \\
& +(2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}} - r\sin\phi\dot{\theta}^2(\sin\phi\hat{\mathbf{r}} + \cos\phi\hat{\boldsymbol{\phi}}) \\
= & (\ddot{r} - r\dot{\phi}^2 - r\sin^2\phi\dot{\theta}^2)\hat{\mathbf{r}} \\
& +(2\sin\phi\dot{\theta}\dot{r} + 2r\cos\phi\dot{\theta}\dot{\phi} + r\sin\phi\ddot{\theta})\hat{\boldsymbol{\theta}} \\
& +(2\dot{r}\dot{\phi} + r\ddot{\phi} - r\sin\phi\cos\phi\dot{\theta}^2)\hat{\boldsymbol{\phi}}.
\end{aligned}
$$

Time derivatives of the unit vectors are

$$
\begin{aligned}
\dot{\hat{\mathbf{r}}} &= \begin{bmatrix} -\sin\theta\sin\phi\,\dot{\theta} + \cos\theta\cos\phi\,\dot{\phi} \\ \cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi} \\ -\sin\phi\,\dot{\phi} \end{bmatrix} = \sin\phi\,\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\hat{\boldsymbol{\phi}} \\
\dot{\hat{\boldsymbol{\theta}}} &= \begin{bmatrix} -\cos\theta\dot{\theta} \\ -\sin\theta\dot{\theta} \\ 0 \end{bmatrix} = -\dot{\theta}\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = -\dot{\theta}(\sin\phi\hat{\mathbf{r}} + \cos\phi\hat{\boldsymbol{\phi}}) \\
\dot{\hat{\boldsymbol{\phi}}} &= \begin{bmatrix} -\sin\theta\cos\phi\,\dot{\theta} - \cos\theta\sin\phi\,\dot{\phi} \\ \cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi} \\ -\cos\phi\,\dot{\phi} \end{bmatrix} = -\dot{\phi}\hat{\mathbf{r}} + \cos\phi\dot{\theta}\hat{\boldsymbol{\theta}}.
\end{aligned}
$$

The curl is

$$
\nabla \times \mathbf{F} = \frac{1}{r\sin\phi}\left[\frac{\partial}{\partial\phi}(\sin\phi F_\theta) - \frac{\partial F_\phi}{\partial\theta}\right]\hat{\mathbf{r}}
$$

$$
+\frac{1}{r}\left[\frac{1}{\sin\phi}\frac{\partial F_r}{\partial\theta} - \frac{\partial}{\partial r}(rF_\theta)\right]\hat{\boldsymbol{\phi}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\phi) - \frac{\partial F_r}{\partial\phi}\right]\hat{\boldsymbol{\theta}}.
$$

The Laplacian is

$$
\begin{aligned}
\nabla^2 &\equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) \\
&= \frac{1}{r^2}\left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\left(\cos\phi\frac{\partial}{\partial\phi} + \sin\right. \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\phi}{r^2\sin\phi}\frac{\partial}{\partial\phi} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}.
\end{aligned}
$$

The vector Laplacian in spherical coordinates is given by

$$\nabla^2 \mathbf{v} = [\ \ ].$$

To express partial derivatives with respect to Cartesian axes in terms of partial
the spherical coordinates,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{bmatrix}$$

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi\,dr - r\sin\theta\sin\phi\,d\theta + r\cos\theta\cos\phi\,d\phi \\ \sin\theta\sin\phi\,dr + r\sin\phi\cos\theta\,d\theta + r\sin\theta\cos\phi\,d\phi \\ \cos\phi\,dr - r\sin\phi\,d\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix} \begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix}.$$

Upon inversion, the result is

$$\begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\frac{\sin\theta}{r\sin\phi} & \frac{\cos\theta}{r\sin\phi} & 0 \\ \frac{\cos\theta\cos\phi}{r} & \frac{\sin\theta\cos\phi}{r} & -\frac{\sin\phi}{r} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

The Cartesian partial derivatives in spherical coordinates are therefore

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$= \cos\theta\sin\phi\frac{\partial}{\partial r} - \frac{\sin\theta}{r\sin\phi}\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta}{r\sin\phi}\frac{\partial}{\partial\theta} + \frac{\sin\theta\cos\phi}{r}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}$$

$$= \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}$$

(Gasiorowicz 1974, pp. 167-168).

The Helmholtz differential equation is separable in spherical coordinates.

SEE ALSO: Azimuth, Colatitude, Great Circle, Helmholtz Differential Equation--Spl
Coordinates, Latitude, Longitude, Oblate Spheroidal Coordinates, Prolate Spher

Coordinates, Zenith

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