# Notes for Quantum Mechanics

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## Lecture 9

Ok. We have a system we are studying - electrons. It looks like they can be characterized by just 2 states, $|+\rangle$ and $|-\rangle$. What are these things? - well they are the eigenkets of $\hat{S}_z$ since

$$\hat{S}_z\,|+\rangle = +\frac{\hbar}{2}\,|+\rangle \quad \text{and } \hat{S}_z\,|-\rangle = -\frac{\hbar}{2}\,|-\rangle \tag{1}$$

and the eigenvalues are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. So they are the eigenkets of an observable $\hat{S}_z$. These two kets are orthogonal because eigenkets of an observable are orthogonal. Now we can use

$\hat{1} = \sum_a |a\rangle\langle a|$ to write

$$\hat{1} = |+\rangle\langle +| + |-\rangle\langle -| \tag{2}$$

Then we can use Lecture 7 eqn (6) $\hat{A} = \sum_i a_i\,|a_i\rangle\langle a_i|$ to write

$$\hat{S}_z = \frac{\hbar}{2}(|+\rangle\langle +| - |-\rangle\langle -|) \quad \text{So we have a way to write } \hat{S}_z \text{ in terms of the base kets} \tag{3}$$

Now lets define a few other things

$$\hat{S}_+ \equiv \hbar|+\rangle\langle -| \quad \text{and} \quad \hat{S}_- \equiv \hbar|-\rangle\langle +| \quad \text{note that these are NOT HERMITIAN} \tag{4}$$

we will see in a bit that

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y \tag{5}$$

$$\text{and } \hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \quad \text{and } \hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-) \tag{6}$$

Now lets see what $S_\pm$ does to the base kets (7)

$$\hat{S}_+|-\rangle = \hbar|+\rangle\langle -|-\rangle = \hbar|+\rangle \quad \hat{S}_+|+\rangle = \hbar|+\rangle\langle -|+\rangle = \hbar|+\rangle 0 = 0 \quad \text{so } \hat{S}_+ \text{ "raises" } |-\rangle \text{ to } |+\rangle \text{ but kills } |+\rangle \tag{8}$$

$$\hat{S}_-|+\rangle = \hbar|-\rangle\langle +|+\rangle = \hbar|-\rangle \quad \hat{S}_-|-\rangle = \hbar|-\rangle\langle +|-\rangle = \hbar|-\rangle 0 = 0 \quad \text{so } \hat{S}_- \text{ "lowers" } |+\rangle \text{ to } |-\rangle \text{ but kills } |-\rangle \tag{9}$$

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Now we know that $|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|S_x;-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$ should be eigenkets of $\hat{S}_x$ with eigenvalues $\pm\frac{\hbar}{2}$

So lets try it $\hat{S}_x\,|S_x;+\rangle = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = \frac{\hbar}{2\sqrt{2}}[|+\rangle\langle -|+\rangle + |+\rangle\langle -|-\rangle + |-\rangle\langle +|+\rangle + |-\rangle\langle +|-\rangle] = \frac{\hbar}{2\sqrt{2}}(|+\rangle + |-\rangle) = \frac{\hbar}{2}\,|S_x;+\rangle$

So $|S_x;+\rangle$ is an eigenket of $\hat{S}_x = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|)$ with an eigenvalue of $\frac{\hbar}{2}$ (10)

We can similarly show that $|S_x;-\rangle$ is an eigenket of $\hat{S}_x$ with an eigenvalue of $-\frac{\hbar}{2}$ (11)

and $\hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-) = \frac{\hbar}{2i}(|+\rangle\langle -| - |-\rangle\langle +|)$ has eigenkets $|S_y;\pm\rangle$ with eigenvalues $\pm\frac{\hbar}{2}$ (12)

Now we can write things in matrices as well using $|+\rangle$ and $\langle -|$ as a basis

$$\hat{X} \doteq \begin{pmatrix} \langle a_1|\hat{X}|a_1\rangle & \langle a_1|\hat{X}|a_2\rangle \\ \langle a_2|\hat{X}|a_1\rangle & \langle a_2|\hat{X}|a_2\rangle \end{pmatrix} \quad \text{so} \quad \hat{S}_z \doteq \begin{pmatrix} \langle +|\hat{S}_z|+\rangle & \langle +|\hat{S}_z|-\rangle \\ \langle -|\hat{S}_z|+\rangle & \langle -|\hat{S}_z|-\rangle \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}$$

$$|\alpha\rangle \doteq \begin{pmatrix} \langle a_1|\alpha\rangle \\ \langle a_2|\alpha\rangle \end{pmatrix} \text{ so } |+\rangle \doteq \begin{pmatrix} \langle +|+\rangle \\ \langle -|+\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |-\rangle \doteq \begin{pmatrix} \langle +|-\rangle \\ \langle -|-\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\langle\alpha| \doteq (\langle\alpha|a_1\rangle \;\; \langle\alpha|a_2\rangle) \quad \text{So } \langle +| \doteq (\langle +|+\rangle \;\; \langle +|-\rangle) = (1\;0) \text{ and } \langle -| \doteq (\langle -|+\rangle \;\; \langle -|-\rangle) = (0\;1) \tag{14}$$

$$\hat{S}_+ \doteq \begin{pmatrix} \langle +|\hat{S}_+|+\rangle & \langle +|\hat{S}_+|-\rangle \\ \langle -|\hat{S}_+|+\rangle & \langle -|\hat{S}_+|-\rangle \end{pmatrix} = \hbar\begin{pmatrix} \langle +|+\rangle\langle -|+\rangle & \langle +|+\rangle\langle -|-\rangle \\ \langle -|+\rangle\langle -|+\rangle & \langle -|+\rangle\langle -|-\rangle \end{pmatrix} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{15}$$

$$\hat{S}_- \doteq \begin{pmatrix} \langle +|\hat{S}_-|+\rangle & \langle +|\hat{S}_-|-\rangle \\ \langle -|\hat{S}_-|+\rangle & \langle -|\hat{S}_-|-\rangle \end{pmatrix} = \hbar\begin{pmatrix} \langle +|-\rangle\langle +|+\rangle & \langle +|-\rangle\langle +|-\rangle \\ \langle -|-\rangle\langle +|+\rangle & \langle -|-\rangle\langle +|-\rangle \end{pmatrix} = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{16}$$

now using (6) we get

$$\hat{S}_x \doteq \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and eigenkets } |S_x;\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle) \doteq \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{17}$$

$$\hat{S}_y \doteq \frac{\hbar}{2i}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\text{and eigenkets } |S_y;\pm\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \pm i\,|-\rangle\right) \doteq \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix} \tag{18}$$

Now lets make sure that the eigenfunctions are really eigenfunctions of the appropriate operator

$S_z$: $\hat{S}_z|+\rangle \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \hat{S}_z|-\rangle \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$S_x$: $\hat{S}_x\,|S_x;+\rangle \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \hat{S}_x\,|S_x;-\rangle \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$


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$$S_y: \ \hat{S}_y \,|\, S_y; +\rangle \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2}\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\hat{S}_y \,|\, S_y; -\rangle \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{\hbar}{2}\,\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ i \end{pmatrix} = -\frac{\hbar}{2}\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

So everything works out nicely

Now lets gather everything together
First Let define some important matrices called the Pauli spin matrices (memorize these!)

$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Then $\hat{S}_z \doteq \frac{\hbar}{2}\hat{\sigma}_z$ $\quad \hat{S}_x \doteq \frac{\hbar}{2}\hat{\sigma}_x$ $\quad \hat{S}_y \doteq \frac{\hbar}{2}\hat{\sigma}_y$ [This is in the $S_z$ representation meaning that the basis kets are the eigenkets of $S_z$ ]

eigenkets of $\hat{S}_z$ are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalues $\pm\frac{\hbar}{2}$ (19)

eigenkets of $\hat{S}_x$ are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$ with eigenvalues $\pm\frac{\hbar}{2}$

eigenkets of $\hat{S}_y$ are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ with eigenvalues $\pm\frac{\hbar}{2}$

$\hat{S}_+ \doteq \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \hat{S}_- \doteq \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

food for thought -

1) Are $|\, S_x; +\rangle$ and $|\, S_y; +\rangle$ orthogonal?

2) are $\hat{x}$ and $\hat{y}$ (the unit vectors in 3 space) orthogonal?

3) are $|\, S_z; +\rangle$ and $|\, S_z; -\rangle$ orthogonal?

4) are $+\hat{z}$ and $-\hat{z}$ orthogonal?


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