Notes for Quantum Mechanics

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Lecture 14

The Heisenberg Uncertainty Priciple

You have probably seen before the uncertainty principle as $\Delta x \Delta p \geq \frac{\hbar}{2}$. This is actually something that is true of many pairs of observables - observables which have operators which do not commute - i.e. that are incompatible. Here is the general form. You will prove this in your homework.

$$\Delta\hat{A}\,\Delta\hat{B} \geq \frac{1}{2}\left|\langle\hat{C}\rangle\right| \text{ where } \hat{C}=[\hat{A},\hat{B}] \quad \text{and } (\Delta\hat{A})^2 \equiv \left\langle\left(\hat{A} - \langle\hat{A}\rangle\right)^2\right\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 \tag{1}$$

Now lets see what this has to say about the spin operators. Recall $[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hbar\hat{S}_k$ specifically $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$

we will get $\Delta\hat{S}_x\,\Delta\hat{S}_y \geq \frac{1}{2}\left|i\hbar\langle\hat{S}_z\rangle\right|$ Lets figure out what this means

First lets figure out what $\Delta\hat{S}_x$ means. Well $(\Delta\hat{S}_x)^2 = \langle\hat{S}_x^2\rangle - \langle\hat{S}_x\rangle^2 = \langle\alpha\,|\,\hat{S}_x^2\,|\,\alpha\rangle - \langle\alpha\,|\,\hat{S}_x\,|\,\alpha\rangle^2$ for some arbitrary state $|\alpha\rangle$

Lets first pick $|\alpha\rangle = |S_x;+\rangle$ then $(\Delta\hat{S}_x)^2 = \langle S_x;+\,|\,\hat{S}_x^2\,|\,S_x;+\rangle - \langle S_x;+\,|\,\hat{S}_x\,|\,S_x;+\rangle^2 =$

$= \langle S_x;+\,|\,\hat{S}_x\hat{S}_x\,|\,S_x;+\rangle - \langle S_x;+\,|\,\frac{\hbar}{2}\,|\,S_x;+\rangle^2 = \langle S_x;+\,|\,\frac{\hbar}{2}\,\frac{\hbar}{2}\,|\,S_x;+\rangle - \left(\frac{\hbar}{2}\right)^2 = \left(\frac{\hbar}{2}\right)^2 - \left(\frac{\hbar}{2}\right)^2 = 0$

Now suppose $|\alpha\rangle = |+\rangle$. Using matrices we get

then $(\Delta\hat{S}_x)^2 = \frac{\hbar^2}{4}\left[(1\ \ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} - \left((1\ \ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)^2\right] =$

$= \frac{\hbar^2}{4}\left[(1\ \ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \left((1\ \ 0)\begin{pmatrix}0\\1\end{pmatrix}\right)^2\right] = \frac{\hbar^2}{4}\left[(1\ \ 0)\begin{pmatrix}1\\0\end{pmatrix} - (0)^2\right] = \frac{\hbar^2}{4}$

so $\Delta\hat{S}_x = \frac{\hbar}{2}$ that is the uncertainty in $\hat{S}_x$ is $\frac{\hbar}{2}$

$(\Delta\hat{S}_y)^2 = \frac{\hbar^2}{4}\left[(1\ \ 0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} - \left((1\ \ 0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)^2\right] = \frac{\hbar^2}{4}\left[(1\ \ 0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\i\end{pmatrix} - \left((1\ \ 0)\begin{pmatrix}0\\i\end{pmatrix}\right)^2\right] =$

$= \frac{\hbar^2}{4}\left[(1\ \ 0)\begin{pmatrix}1\\0\end{pmatrix} - (0)^2\right] = \frac{\hbar^2}{4}$

so $\Delta\hat{S}_y = \frac{\hbar}{2}$ and $\langle\hat{S}_z\rangle = \langle +\,|\,\hat{S}_z\,|\,+\rangle = \frac{\hbar}{2}$ so this gives $\frac{\hbar}{2}\,\frac{\hbar}{2} \geq \frac{1}{2}\left|i\hbar\,\frac{\hbar}{2}\right|$ which gives us just $\frac{\hbar}{4} \geq \frac{\hbar}{4}$ which is true

Next we will get into position and momentum operators, where the Heiseberg Uncertainty principle will appear in its usual form.

