Notes for Quantum Mechanics

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Lecture 10

Measurement

Dirac- "A measurement always causes the system to jump into an eigenstate of the dynamical variable being measured"

$\Longrightarrow$ A measurement usually changes the system

A measurement is a filtering process. if $|\alpha\rangle = \sum_{a'} |a'\rangle \langle a' | \alpha\rangle$ remembering that $|a'\rangle$ are the eigenkets of $\hat{A}$ then (1)

$$|\alpha\rangle \xrightarrow{\text{measurement of } \hat{A}} |a'\rangle$$

or in words - if a system is in a state $|\alpha\rangle$ and a measurement is made of the observable $\hat{A}$ then the system will jump into one of the eigenkets of $\hat{A}$ i.e. $|a'\rangle$

Postulate: (which cannot be proven) - The probability for a system $|\alpha\rangle$ to be in a state $|a'\rangle$ is $|\langle a' | \alpha\rangle|^2$ (2)

note: a pure ensemble is a collection of identically prepared systems all in a state $|\alpha\rangle$ (we will see what this means later)

We define an expectation value of the operator $\hat{A}$ for a system $|\alpha\rangle$ as $\langle \hat{A}\rangle_\alpha \equiv \langle \alpha | \hat{A} | \alpha\rangle$ (3)

Let's look at this

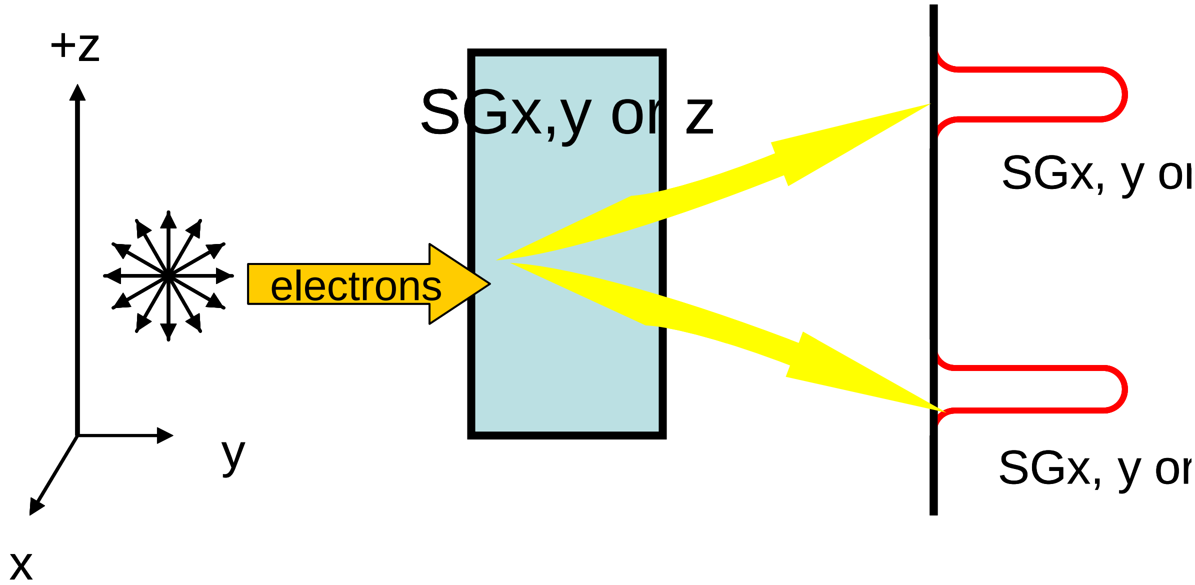
$$\langle \hat{A}\rangle = \langle \alpha | \hat{A} | \alpha\rangle = \sum_{a'} \sum_{a''} \langle \alpha | a''\rangle \langle a'' | \hat{A} | a'\rangle \langle a' | \alpha\rangle = \sum_{a'} a' \, |\langle a' | \alpha\rangle|^2 \tag{4}$$

where $|\langle a' | \alpha\rangle|^2$ is the probability of obtaining a' from the measurement of $\hat{A}$ so this definition of an expectation value makes sense in that it weights each value by the probability

Remember that an eigenvalue is NOT an expectation value - for instance the expectation values for $\hat{S}_z$ are $\pm \frac{\hbar}{2}$ but $\langle \hat{S}_z\rangle = 0.2\hbar$ for example.

Back to the Stern Gerlach Experiment

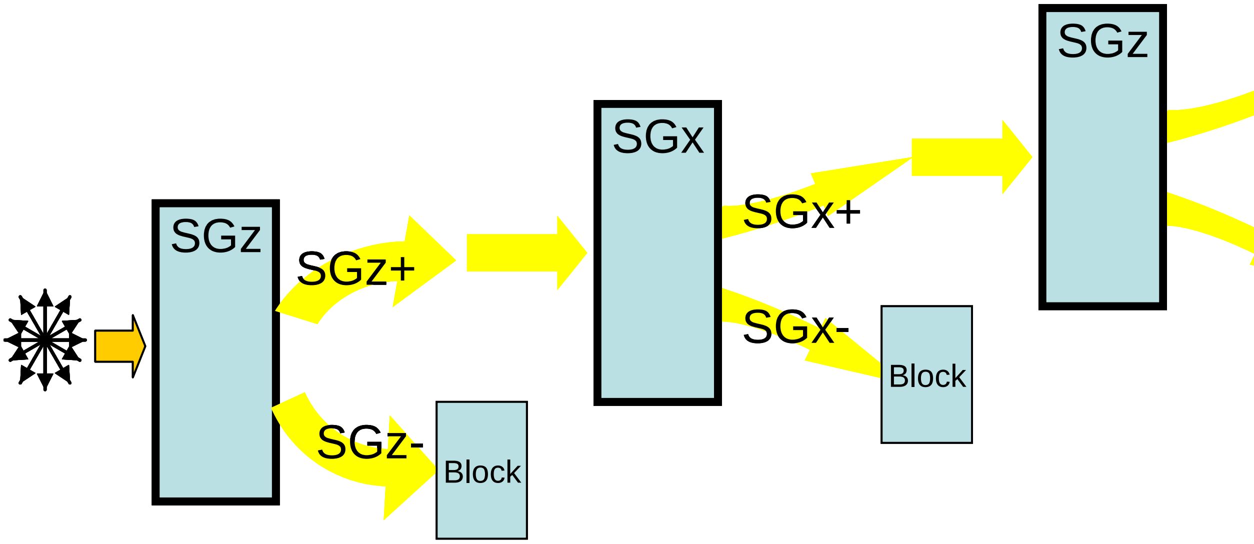
Now lets see if we can't make some more sense of the SG experiment. Remember that we took a beam of electrons which were presumably pointing in any random direction and put it through a SGz apparatus. It appeared as if the incoming beam were actually pointing in only the +z or -z directions. We then put it through an SGx apparatus, and it looked like the incoming beam were only pointing in the +x or -x direction. The same thing happened with a SGy apparatus. How can this be? It seems to be contradictory.



Dirac's idea of measurement seems to answer it. Passing the beam through the SGz apparatus is making a measurement of S_z! Once you do that, Dirac says it "jumps" into an eigenstate of S_z (i.e. SGz+ or SGz-). Similarly if you make a measurement of S_x the electrons "jump" into an eigenstate of SGx.

Now this is sort of crazy. You can think of taking a single electron and asking what it is doing. So if you take an electron and measure S_z you will get either SGz + or SGz-. But then if you take that SAME electron and measure SGx, it will either be SGx+ or SGx-.

In fact - this now explains what was going on when we regenerated the SGz- component of the beam in the following experiment.



If we took a SGz+ beam and then measured SGx, it then jumped into an eigenstate and "forgot" its SGzness. When we then measured SGz again, it did not "know" that SGz- was gone before and gave us 50% SGz+ and SGz-. Now this language of "forgetting" is not really accurate. The language of kets which represent the state is really what is "real". It tells us that our common notions of what an electron is, are not correct. The electrons spin is really represented by this beast we call a ket - at least the ket, can tell us what happens in a real experiment - whereas our common notions will lead us astray.

Expectation values

Remember we defined the expectation value of an operator as $\langle \hat{A} \rangle_\alpha \equiv \langle \alpha \mid \hat{A} \mid \alpha \rangle$. Lets think about the SG experiment.

First lets just find the expectation value of a beam of $|+\rangle$

$|\alpha\rangle = |+\rangle$

$\langle \hat{S}_z \rangle_\alpha \equiv \langle \alpha \mid \hat{S}_z \mid \alpha \rangle = \langle + \mid \hat{S}_z \mid + \rangle = \frac{\hbar}{2}$ good! makes sense

Next find the expectation value of a beam which is formed by mixing an equal amount of $|+\rangle$ and $|-\rangle$ so

$|\alpha\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$\langle \hat{S}_z \rangle_\alpha \equiv \langle \alpha \mid \hat{S}_z \mid \alpha \rangle = \frac{1}{2}\left(\langle + | + \langle - |\right) \hat{S}_z \left(|+\rangle + |-\rangle\right) = \frac{1}{2}\left(\langle + | + \langle - |\right) \frac{\hbar}{2}\left(|+\rangle - |-\rangle\right) = \frac{\hbar}{4}(1-1) = 0$ as might be expected

Suppose we make abeam which is 3 parts $|+\rangle$ and 1 part $|-\rangle$. So if we correctly normalize this we get

$|\alpha\rangle = \frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}|-\rangle$. Lets now find the expectation value of S_z for this

$$\langle \hat{S}_z \rangle_\alpha \equiv \langle \alpha \mid \hat{S}_z \mid \alpha \rangle = \left(\frac{\sqrt{3}}{2}\langle + | + \frac{1}{2}\langle - |\right) \hat{S}_z \left(\frac{\sqrt{3}}{2}|+\rangle + \frac{1}{2}|-\rangle\right) =$$

$$\left(\frac{\sqrt{3}}{2}\langle + | + \frac{1}{2}\langle - |\right) \frac{\hbar}{2} \left(\frac{\sqrt{3}}{2}|+\rangle - \frac{1}{2}|-\rangle\right) = \frac{\hbar}{2}\left(\frac{3}{4} - \frac{1}{4}\right) = \frac{\hbar}{4}$$

Let's find the expectation value of S_x for a pure beam of $|+\rangle$ so $|\alpha\rangle = |+\rangle$

There are two ways we can do this. The first way is that we could write the $|+\rangle$ ket in terms of the $|S_x;+\rangle$ and $|S_x;-\rangle$ kets.

We could also write the $\hat{S}_x$ operator in terms of $|+\rangle$ and $|-\rangle$. Lets do it both ways.

First lets write $|+\rangle = \frac{1}{2}\,(|S_x;+\rangle + (|S_x;-\rangle)$ from lecture 8

then $|\alpha\rangle = |+\rangle = \frac{1}{\sqrt{2}}\,(|S_x;+\rangle + (|S_x;-\rangle)$

$\langle\hat{S}_x\rangle_\alpha \equiv \left\langle \alpha \mid \hat{S}_x \mid \alpha \right\rangle = \frac{1}{2}\,(\langle S_x;+| + (\langle S_x;-|)\;\;\hat{S}_x\;\;(|S_x;+\rangle + (|S_x;-\rangle) = \frac{1}{2}\,(\langle S_x;+| + (\langle S_x;-|)\,\frac{\hbar}{2}\,(|S_x;+\rangle - (|S_x;-\rangle) = \frac{\hbar}{4}\,(1-1) = 0$

This matches our expectation that taking a SGz+ beam and measuring SGx splits into 1/2 each of SGx+ and SGx- so the average is zero.

Now lets do it the second way. We can write $\hat{S}_x = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|\,)$ from lecture 9 and $|\alpha\rangle = |+\rangle$

$\langle\hat{S}_x\rangle_\alpha \equiv \left\langle \alpha \mid \hat{S}_x \mid \alpha \right\rangle = \left\langle + \mid \hat{S}_x \mid + \right\rangle = \left\langle + \,\middle|\, \frac{\hbar}{2}\,(|+\rangle\,\langle -| + |-\rangle\,\langle +|) \,\middle|\, + \right\rangle = \frac{\hbar}{2}(\langle + | + \rangle\,\langle - \mid + \rangle + \langle + \mid - \rangle\,\langle + \mid + \rangle) = 0$

So we get the same answer either way as we might have thought.