# The distribution function of the event-by-event average $p_T$ for statistically independent emission

M. J. Tannenbaum [1]

*Brookhaven National Laboratory, Upton, NY 11973-5000 USA*

**Abstract**

For statistically independent emission, a closed form of the distribution function for $M_{p_T}$, the event-by-event average $p_T$, can be obtained. A recent NA49 measurement satisfies this conditions and a distribution is obtained which is in excellent agreement with the NA49 measurement.

*Keywords:* Relativistic heavy-ion collisions; event-by-event analyses; distributions; fluctuations

Recently, the NA49 experiment [1] has presented a measurement of the distribution of $M_{p_T}$, the event-by-event average transverse momentum, for central Pb+Pb collisions at 158 GeV per nucleon. A total of 98426 events was recorded with an average of $270.13 \pm 0.07$ particles per event and an r.m.s. deviation about the average of $23.29 \pm 0.05$ particles. NA49 make several important observations, including: the distribution of $M_{p_T}$ is approximately Gaussian; a mixed event sample reproduces the actual data to high precision; the individual samples, $p_{T_i}$, on a given event are compatible with being statistically independent. It is precisely for such conditions that an analytical formula for the distribution in $M_{p_T}$ can be obtained. The result is shown in Fig. 1, in excellent agreement with the NA49 measurement [1]. The derivation follows.

In the theory of probability and statistics, a statistic is a quantity computed entirely from the sample, i.e. a statistic is any function of the observed sample values. Two of the most popular statistics are the sum and the average:

$$S_n \equiv \sum_{i=1}^{n} x_i \tag{1}$$

$$\bar{x}_{(n)} \equiv \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n} S_n \tag{2}$$

[1] Supported by the U.S. Department of Energy under contract DE-AC02-98CH10886.

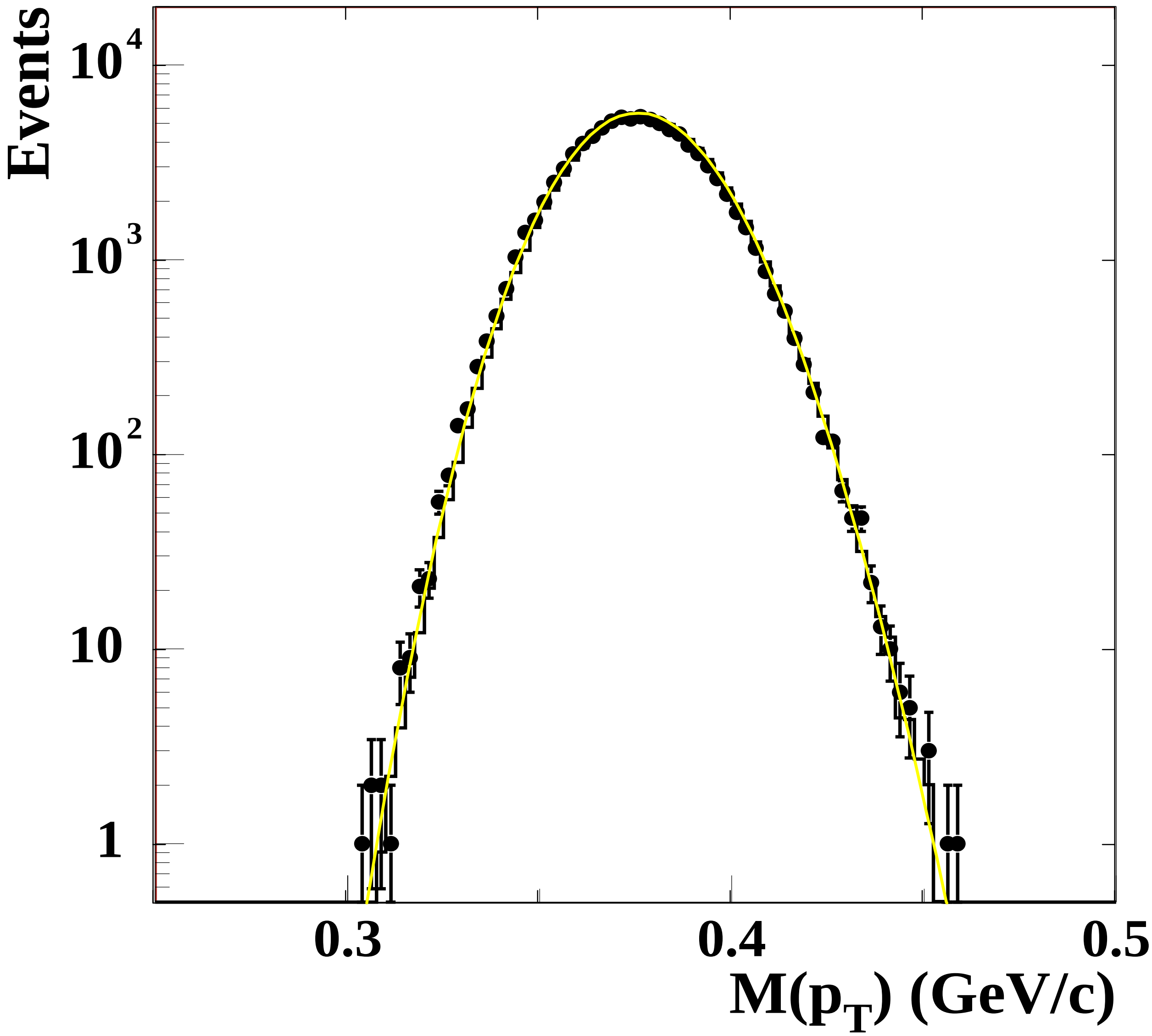


Fig. 1. Full distribution in $M_{p_T}$ (light line) compared to NA49 measurement (filled points) and mixed event distribution (histogram). See reference [1] for details of the measurement. Details of the present calculation are given in the text.

where the $x_i$ are $n$ samples from a the same population or probability density function, $f(x)$. From the theory of mathematical statistics[2], the probability distribution of a random variable $S_n$, which is itself the sum of $n$ independent random variables with a common distribution $f(x)$:

$$S_n = x_1 + x_2 + \cdots + x_n \tag{3}$$

is given by $f_n(x)$, the $n$-fold convolution of the distribution $f(x)$:

$$f_n(x) = \int_0^x dy\, f(y)\, f_{n-1}(x-y) \qquad . \tag{4}$$

The mean, $\mu_n = \langle S_n \rangle$, and standard deviation, $\sigma_n$, of the $n$-fold convolution obey the familiar rule

$$\mu_n = n\mu \qquad \sigma_n = \sigma\sqrt{n} \quad , \tag{5}$$

where $\mu$ and $\sigma$ are the mean and standard deviation of the distribution $f(x)$. A complementary case is that of a random variable $Z_n$, which is the sum of $n$ random variables with a common distribution $f(x)$—which are themselves 100% correlated—for example:

$$Z_n = x + x + \cdots + x = nx \qquad . \tag{6}$$

This is just a scale transformation. The behavior of the mean and the standard deviation for a scale transformation is $\mu \to n\mu$, $\sigma \to n\sigma$ , which is quite different than the behavior of the standard deviation under convolution (Eq. 5).

The Gamma distribution is an example of a probability density function (pdf) which has particularly simple properties under convolutions and scale transformations. The Gamma distribution is a function of a continuous variable $x$ and has paramters $p$ and $b$

$$f(x) = f_\Gamma(x,p,b) = \frac{b}{\Gamma(p)}(bx)^{p-1}e^{-bx} \tag{7}$$

where

$$p > 0, \quad b > 0, \quad 0 \le x \le \infty$$

$\Gamma(p) = (p-1)!$ if $p$ is an integer, and $f(x)$ is normalized, $\int_0^\infty f(x)dx = 1$. The mean and standard deviation of the distribution are

$$\mu \equiv \langle x \rangle = \frac{p}{b} \qquad \sigma \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{\sqrt{p}}{b} \qquad \frac{\sigma^2}{\mu^2} = \frac{1}{p} \qquad . \tag{8}$$

The $n$-fold convolution of the Gamma distribution (Eq. 7) is simply given by the function

$$f_n(x) = \frac{b}{\Gamma(np)}(bx)^{np-1}e^{-bx} = f_\Gamma(x, np, b) \tag{9}$$

i.e. $p \to np$ and $b$ remains unchanged. Note that the mean and standard deviation of Eq. 9

$$\mu_n = \frac{np}{b} \qquad \sigma_n = \frac{\sqrt{np}}{b} \qquad \frac{\sigma_n}{\mu_n} = \frac{1}{\sqrt{np}} \tag{10}$$

when compared to Eq. 8 explicitly obey Eq. 5. The result of a scale transformation $x \to nx$ for a Gamma distribution (Eq. 7) is simply $b \to b/n$, with $p$ remaining unchanged. To summarize, the $n$-th convolution of the Gamma distribution $f_\Gamma(x, p, b)$ is $f_\Gamma(x, np, b)$; the scale transformation $x \to nx$ of $f_\Gamma(x, p, b)$ is $f_\Gamma(x, p, b/n)$.

The principal advantage of the Gamma distribution for the present problem is that it is one of the standard representations of the inclusive single particle $p_T$ distribution:

$$\frac{d\sigma}{p_T dp_T} = b^2 e^{-bp_T} \tag{11}$$

$$\frac{d\sigma}{dp_T} = b^2 p_T e^{-bp_T} \qquad . \tag{12}$$

Clearly, Eq.s 11, 12 represent a Gamma distribution with $p = 2$, $\langle p_T \rangle = 2/b$, where typically $b = 6$ $(\mathrm{GeV/c})^{-1}$ for p-p collisions. The 'inverse slope parameter' $1/b$ is sometimes referred to as the 'Temperature parameter'.

The NA49 [1] event-by-event variable, which they denote $M_{p_T}$, is just the event-by-event average transverse momentum

$$M_{p_T} = \overline{p_T}_{(n)} = \frac{1}{n}\sum_{i=1}^{n} p_{T_i} = \frac{1}{n} E_{Tc} \tag{13}$$

where

$$E_{Tc} = \sum_{i=1}^{n} p_{T_i} \qquad , \tag{14}$$

$p_{T_i}$ is the magnitude of the transverse momentum for the $i$-th charged particle in an event and $n$ is the number of accepted charged particles for the event. Cognoscente will recognize Eq. 14 as the event-by-event variable '$E_T$' for charged particles [3]. The solution for the probability distribution function for $M_{p_T}$ (Eq. 13) follows from Eqs. 3 and 4 if the particles on a given event are independently emitted. Taking the $p_T$ distribution for a single particle as Eq. 7, $f_\Gamma(x, p, b)$, the $E_{Tc}$ distribution for $n$ independent particles is the $n$-th convolution, $f_\Gamma(E_{Tc}, np, b)$. Finally, the relation between $E_{Tc}$ and $M_{p_T} = E_{Tc}/n$ is just a scale change $x \to x/n$ so the final distribution for $M_{p_T}$ is $f_\Gamma(M_{p_T}, np, nb)$,

$$f(y) = f_\Gamma(y, np, nb) = \frac{nb}{\Gamma(np)}(nby)^{np-1} e^{-nby} \tag{15}$$

where the variable $y$ represents $M_{p_T}$. The mean and standard deviation of the distribution in $M_{p_T}$ are:

$$\mu_{M_{p_T}} = \frac{np}{nb} = \frac{p}{b} = \langle p_T \rangle \qquad \sigma_{M_{p_T}} = \frac{\sqrt{np}}{nb} = \frac{\sigma_{p_T}}{\sqrt{n}} \qquad \frac{\sigma_{M_{p_T}}}{\mu_{M_{p_T}}} = \frac{1}{\sqrt{np}} \quad . \tag{16}$$

Strictly speaking, this is valid for fixed $n$, but we ignore this for the moment.

NA49 [1] give values of $\langle p_T \rangle = 0.37675 \pm 0.00006$ GeV/c, $\sigma_{p_T} = 0.2822 \pm 0.0001$ GeV/c for the inclusive average over all accepted particles and all events. These two moments are sufficient to obtain the $b$ and $p$ parameters of the semi-inclusive single particle $p_T$ distribution, assumed to be of the form $f_\Gamma(x, p, b)$ (Eqs. 7, 8):

$$p = \frac{\langle p_T \rangle^2}{\sigma_{p_T}^2} = 1.782 \pm 0.001 \tag{17}$$

$$b = \frac{\langle p_T \rangle}{\sigma_{p_T}^2} = 4.730 \pm 0.002 \quad (\mathrm{GeV/c})^{-1} \qquad . \tag{18}$$

The derived $p$ parameter of 1.782 for the semi-inclusive $p_T$ distribution does not equal to 2, as expected from Eq. 12, but is reasonably close considering that the quoted NA49 averages and standard deviations are for a truncated range, $0.005 < p_T < 1.5$ GeV/c, and not for the full distribution. Using the predicted distribution for $M_{p_T}$ for the case of independent emission (Eqs. 15, 16), the standard deviation of $M_{p_T}$ can be computed from the number of particles per event $n$ (assumed fixed at the average) and the parameters $b$ and $p$ of the semi-inclusive distribution:

$$\frac{\sigma_{M_{p_T}}}{\mu_{M_{p_T}}} = \frac{1}{\sqrt{np}} = 4.56\% \qquad . \tag{19}$$

This value of 4.56% for $\sigma_{M_{p_T}}/\mu_{M_{p_T}}$ is in excellent agreement with, although slightly smaller than the measured value of 4.65±0.01%, leaving scant room for possible correlation effects which would destroy the statistical independence.

The distribution in $M_{p_T}$ (Eq. 15) for a fixed value of $n = \langle n \rangle = 270.13$ is shown in Fig. 2 using the values of $p$ and $b$ (Eqs. 17, 18) derived from the semi-inclusive $p_T$ distribution where the plot is normalized to 98426 events (dashed curve). The full distribution in $M_{p_T}$ from Fig. 1, properly averaged over the variation in $n$, is also shown (solid curve). The distribution with fixed $n = \langle n \rangle$ is slightly narrower and barely distinguishable by eye from the full $M_{p_T}$ distribution and both are strikingly in agreement with the NA49 measurement (recall Fig. 1). The detailed difference between a Gamma distribution and a Gaussian is illustrated in Fig. 3 where a Gaussian with the same mean and $\sigma$ as the Gamma distribution with fixed $n = \langle n \rangle$ is superimposed onto Fig. 1 (see Fig. 3). Perhaps very trained eyes will note that the upper edges of the Gamma

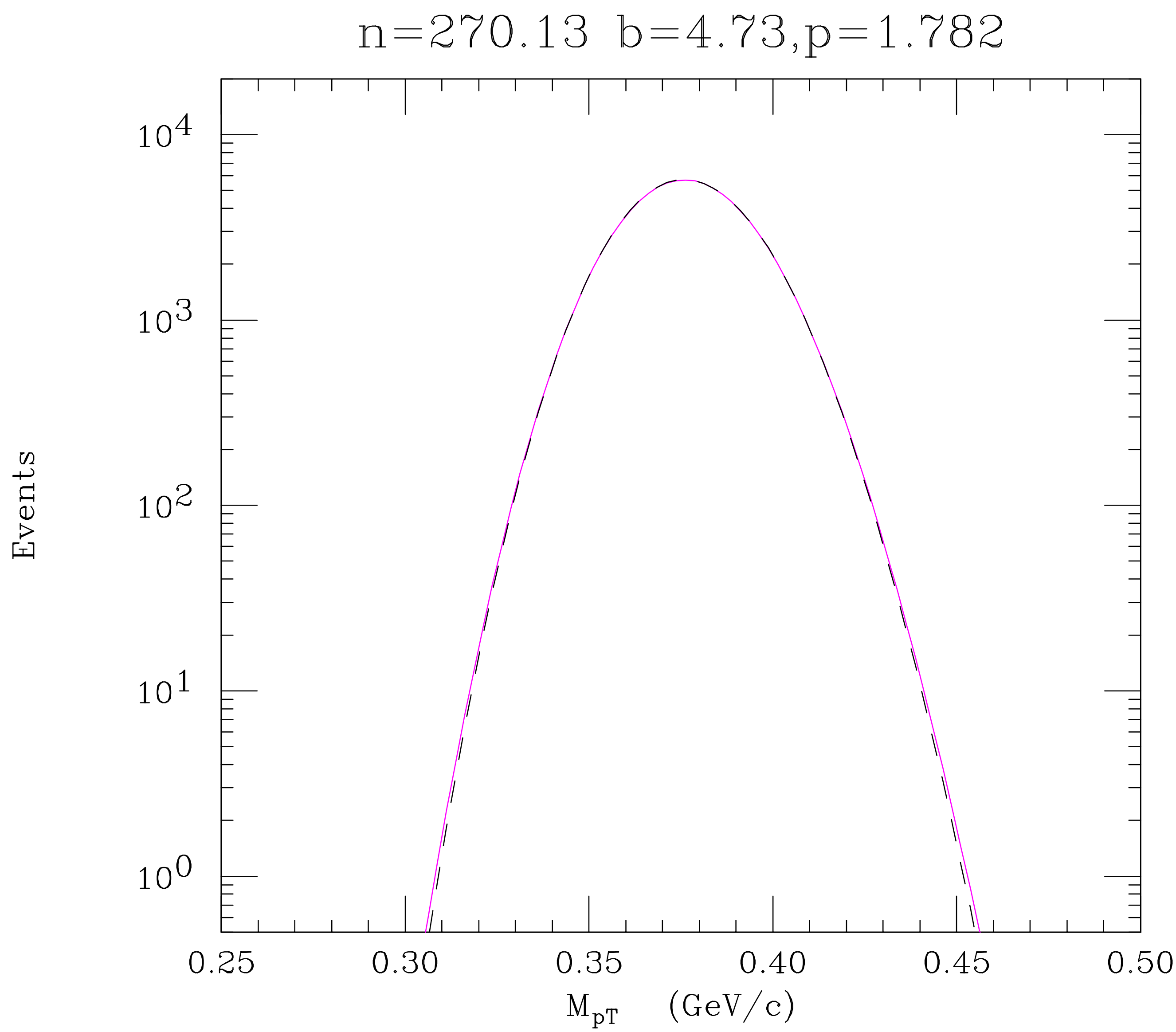


Fig. 2. The calculated $M_{p_T}$ distribution for independent emission of a fixed number $n = 270.13$ particles on an event (Eq. 15) using the parameters $b = 4.730$ GeV$^{-1}$ and $p = 1.782$ derived from the mean and standard deviation of the semi-inclusive $p_T$ distribution, averaged over all events, (dashed line) compared to the full distribution in $M_{p_T}$, properly averaged over the variation in $n$, from Fig. 1 (solid line). Following NA49 [1], the distribution is given in events per 0.0025 GeV/c bin, normalized to a total number of 98426 events.

distribution curve and the data are straight lines (on the semi-log plot) for $M_{p_T} \gtrsim 0.42$, illustrating the asymptotic exponential slope characteristic of a Gamma distribution, while the Gaussian is parabolic in shape and falls below the data. This suggests an additional test for statistical independence of $n$ particles emitted on a given event: the asymptotic 'inverse slope parameter' of the distribution in $M_{p_T}$, $1/nb$, should be $1/n$ times the semi-inclusive inverse slope parameter, $1/b$. It is also clear that a statistical test to reject one of the two curves would require an additional order of magnitude in data.

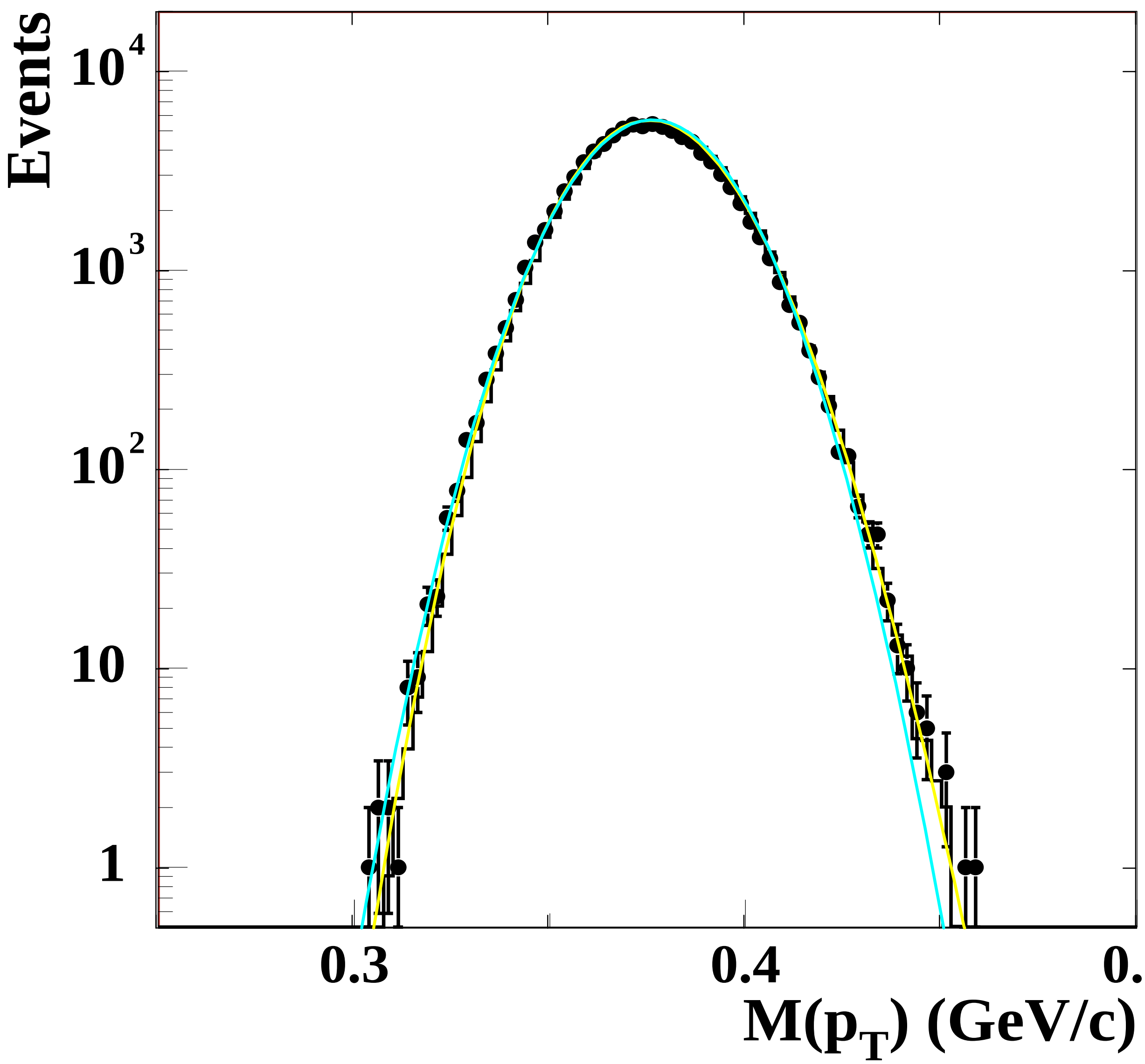


Fig. 3. From Fig. 1, full distribution in $M_{p_T}$ (light line) compared to NA49 measurement (filled points) and mixed event distribution (histogram) compared to a Gaussian (darker line) with the same mean and $\sigma$ as the Gamma distribution with fixed $n = 270.13$ from Fig. 2.

The derivation of the full distribution, where the variation of $n$ is taken into account, is straightforward but requires significant additional computation. First, the sensitivity of the distribution of $M_{p_T}$ to different values of fixed $n$ is shown in Fig 4 for three fixed values of $n$: $n = \langle n \rangle = 270.13$, and $n$=240, 300, each approximately 1.5 standard deviations from the mean multiplicity per event. The distributions are very similar but can be clearly distinguished. The

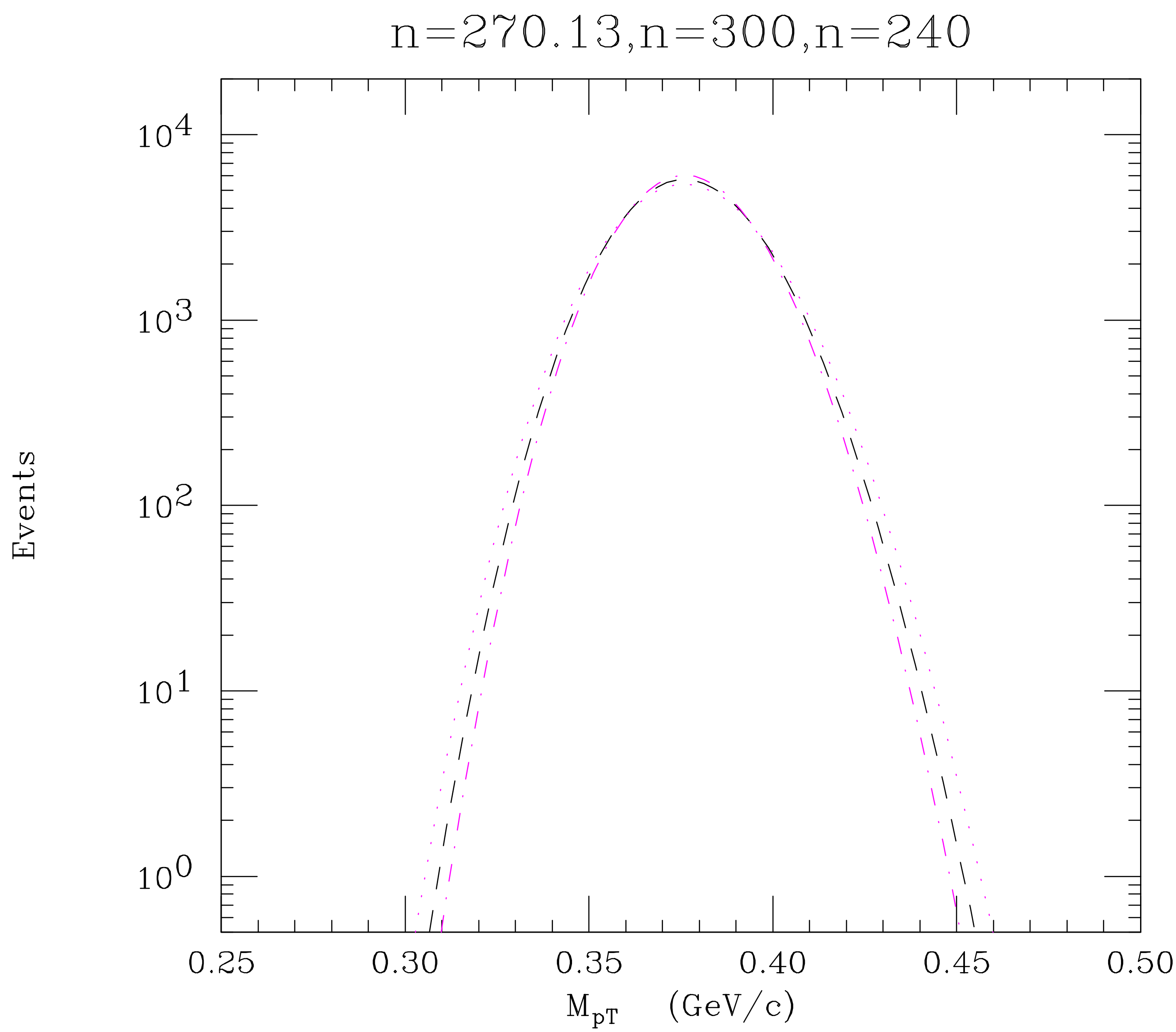


Fig. 4. Distribution in $M_{p_T}$ for 3 fixed values of $n$: $n = \langle n \rangle = 270.13$ (dashes), $n = 300$ (dotdash), $n = 240$ (dots).

full distribution in $M_{p_T}$, averaged over the NA49 multiplicity distribution for central collisions, is obtained by assuming that the multiplicity distribution is Negative Binomial [4,5]:

$$P(n) = f_{\rm NBD}(n, 1/k, \mu) = \frac{(n+k-1)!}{n!(k-1)!} \, \frac{(\frac{\mu}{k})^n}{(1+\frac{\mu}{k})^{n+k}} \tag{20}$$

where $P(n)$ is normalized for $0 \leq n \leq \infty$, $\mu \equiv \langle n \rangle$, and the standard deviation is:

$$\sigma = \sqrt{\mu(1+\frac{\mu}{k})} \qquad \frac{\sigma^2}{\mu^2} = \frac{1}{\mu} + \frac{1}{k} \qquad . \tag{21}$$

The NBD parameter $1/k$ for NA49 central Pb+Pb collisions can be calculated from the quoted $\langle n \rangle = 270.13 \pm 0.07$ and $\sigma = 23.39 \pm 0.05$, yielding:

$$\frac{1}{k} = \frac{\sigma^2}{\langle n \rangle^2} - \frac{1}{\langle n \rangle} = 0.00373 \pm 0.00002 \qquad . \tag{22}$$

The distribution in $M_{p_T}$ for fixed $n$, $f_\Gamma(y, np, nb)$, is then averaged over the central collision multiplicity distribution, $f_{\rm NBD}(n, 1/k, \langle n \rangle)$, to obtain:

$$f(y) = \sum_{n=n_{\rm min}}^{n_{\rm max}} f_{\rm NBD}(n, 1/k, \langle n \rangle)\, f_\Gamma(y, np, nb) \qquad . \tag{23}$$

This is the full distribution in $M_{p_T}$ for NBD distributed event-by-event multiplicity, with Gamma distributed semi-inclusive $p_T$ spectrum, assuming statistically independent emission of particles on each event. It depends on the 4 semi-inclusive paramters $\langle n \rangle$, $1/k$, $b$ and $p$. This distribution was shown in Figs 1, 2, 3, where values of $n_{\rm min} = 200$ and $n_{\rm max} = 340$ were used. As noted previously, agreement with the NA49 data is exceptional.

The distribution in the event-by-event average transverse momentum, $M_{p_T}$, measured by NA49 [1] for Pb+Pb central collisions has been calculated for the case where the individual particles on a given event are emitted statistically independently. A similar analysis could not be performed for correlated emission of particles since no simple analogy of Eqs. 3, 4, 6 exists for this case. The parameters of the event-by-event distribution are obtained from the means and standard deviations of the semi-inclusive $p_T$ and central multiplicity distributions quoted by NA49, yielding a distribution in $M_{p_T}$ which appears to reproduce very well the measured distribution. The statistical independence of the emission of particles on a given event can be seen by the relationship of the standard deviation of the $M_{p_T}$ distribution to its mean (Eqs. 16, 19) in comparison to the semi-inclusive quantities (Eq. 8), as noted by NA49 [1]. A new test suggested by the present work is that the asymptotic slope of the $M_{p_T}$ distribution (Eq. 15) should be $n$ times the asymptotic slope of the semi-inclusive $p_T$ distribution (Eq. 12).

The present derivation has much in common with the techniques used for $E_T$ distributions [3] as both make use of the elegant properties of Gamma distributions. However there is a significant difference. Analyses of $E_T$ distributions in wounded nucleon models start with a measured elementary multiparticle $E_T$ distribution in p-p or p+A collisions, represented as a Gamma distribution, and reconstruct the $E_T$ distribution for A+A collisions as the weighted sum of convolutions of elementary Gamma distributions, where the weights are given by a model. Any correlations in single particle emission for the elementary p-p or p+A process are taken into account by using the measured multiparticle $E_T$ distribution. No attempt is made to derive the elementary multiparticle $E_T$ distribution from the single particle $p_T$ spectrum because of the large correlations in p-p and p+A collisions. The present analysis is quite different: the multiparticle distribution in event-by-event $M_{p_T}$ for Pb+Pb central collisions is derived from the semi-inclusive single particle $p_T$ spectrum for the same sample of events by assuming statistically independent emission of particles. No model is involved, just a straightforward statistical analysis. If there were any correlations in transverse momentum emission of particles on a given event, or two distinct classes of events with different single particle transverse momentum spectra, the derived distribution would disagree with the measurement .

Another classical example in nuclear physics which perhaps has more in common with the distribution of event-by-event average transverse momentum is the distribution in the time interval between every $n$th count of radioactive decay, where the probability is exponential for the time interval between counts[6,7]. Since an exponential is just a Gamma distribution (Eq. 7) with $p = 1$, the distribution for the time $x$ between $n$ counts is given by Eq. 9 with $p = 1$ and $b = \lambda$, the normalized probability of decay per unit time. The reduction, or 'regularization', of the relative fluctuations of the counting rate distribution of a radioactive source when pre-scaled by $n$ illustrates essentially the same effect as the event-by-event average transverse momentum distribution for independent emission of $n$ particles per event.

## References


[1] NA49 Collaboration, H. Appelshäuser, *et al., Phys. Lett.* **B459** (1999) 679.

[2] W. Feller, *An Intorduction to Probability Theory and Its Applications.* Wiley, New York, 1966.

[3] See, for example, M. J. Tannenbaum, *Int. J. Mod. Phys.* **A4** (1989) 3377, and references therein.

[4] UA5 collaboration, G. J. Alner, *et al., Phys. Lett.* **160B** (1985) 193; see also UA5 collaboration, G. J. Alner, *et al., Physics Reports* **154** (1987) 247, and references therein.

[5] T. Abbott, *et al*, E802 Collaboration, *Phys. Rev.* **C52** (1995) 2663.

[6] D. Halliday, *Introductory Nuclear Physics.* Wiley, New York, 1955, p. 211; A. C. Melissinos, *Experiments in Modern Physics.* Academic Press, New York, 1966. pp. 484-487.

[7] J. Rainwater and C. S. Wu, *Nucleonics* **1** (1947) 62.