AdS/CFT correspondence and the Quark-Gluon Plasma

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Problem with strong coupling

- QCD is asymptotically free theory, α_s is small at large momentum scale
- Perturbative expansion of QCD thermodynamics is not well behaved for realistic temperatures.
- For thermodynamics we can use lattice, resummation techniques (Blaizot's talk).
- Kinetic coefficients are almost impossible to extract from lattice data

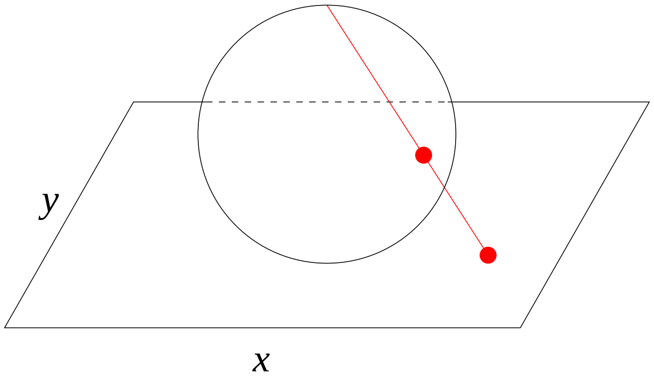
String theory as a tool

- From the work in the late 90's, a remarkable connection has appeared between a certain string theory in certain curved spacetimes and certain gauge theories in flat (3+1) dimensional space
- This is originally called the anti de-Sitter / Conformal field theory correspondence (AdS/CFT), Maldacena's conjecture.
- Extensions of this idea shed fresh new light on the problem of quark confinement

AdS space: 2D illustration

Sphere in projective coordinates:

$$ds^2 = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}$$



This is a space with constant positive curvature.

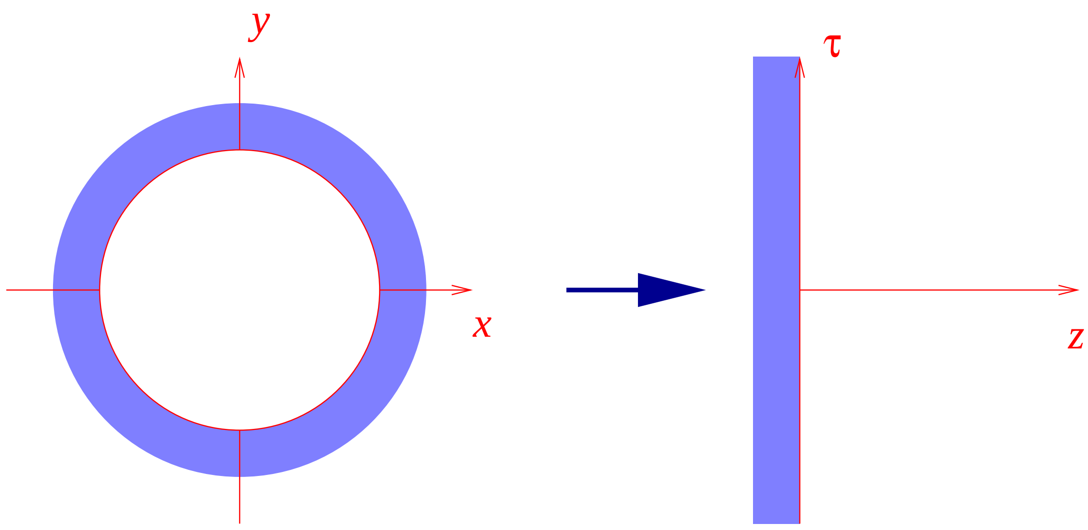
To make a space with constant negative curvature, we change signs in the denominator.

The result is Euclidean AdS_2 space:

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

AdS space (continued)

Now we perform conformal transformation:



$$z + i\tau = \frac{1 - x - iy}{1 + x + iy} \Rightarrow ds^2 = \frac{dz^2 + d\tau^2}{z^2}$$

In more than two dimensions, Minkowski signature:

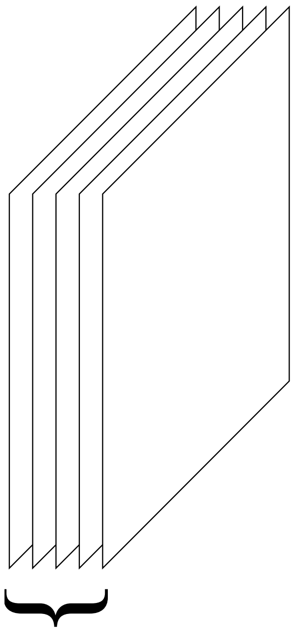
$$ds^2 = \underbrace{\frac{R^2}{z^2}}_{\text{warp factor}} \left(-dt^2 + d\vec{x}^2 + dz^2\right)$$

The Gauge/Gravity Duality

Stack of N D3-branes in type IIB string theory: described in two different pictures:

As a quantum field theory of degrees of freedom on the branes: $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

As string theory on a the curved spacetime (induced by the matter density on the branes)



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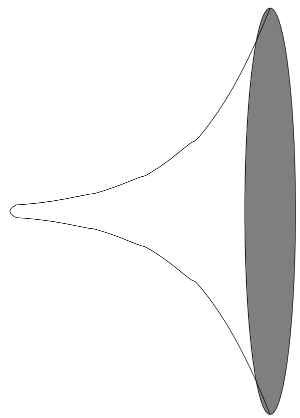


The limit of infinitely strong coupling in gauge theory is the limit when string theory becomes Einstein's general relativity

The two sides of the correspondence

- The $\mathcal{N} = 4$ SYM theory:
  - contains gauge field, 4 Weyl fermions, 6 real scalars.
  - Zero beta function: g is a parameter of the theory (compared to QCD: $\Lambda_{\rm QCD}$ is a parameter). Also N_c.
- Type II B string theory:
  - contains a finite number of massless fields (graviton, dilatons, forms), infinite number of massive string excitations
  - two parameters: inverse string tension $\alpha' = l_{\rm st}^2$, and string coupling $g_{\rm st}$
  - on AdS$_5 \times$ S^5 background

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2} d\Omega_5^2$$

Mapping of parameters

The AdS/CFT correspondence between parameters of two theories:

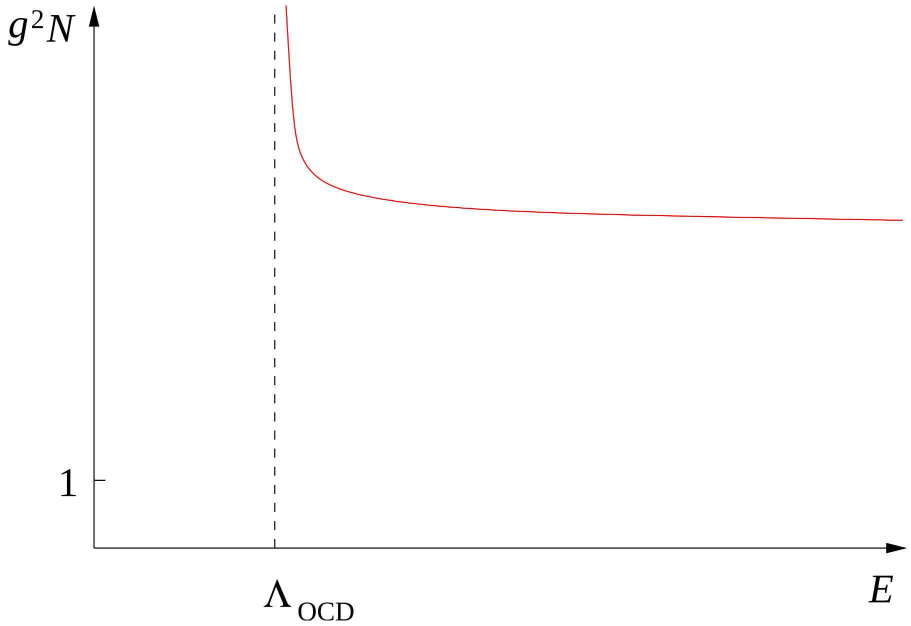
$$g^2 = 4\pi g_{\mathrm{st}}$$

$$g^2 N_c = \left(\frac{R}{l_{\mathrm{st}}}\right)^4$$

AdS/CFT is useful in the limit $g \to 0$, $g^2 N_c \to \infty$: string theory becomes classical supergravity. Allows analytical treatment.

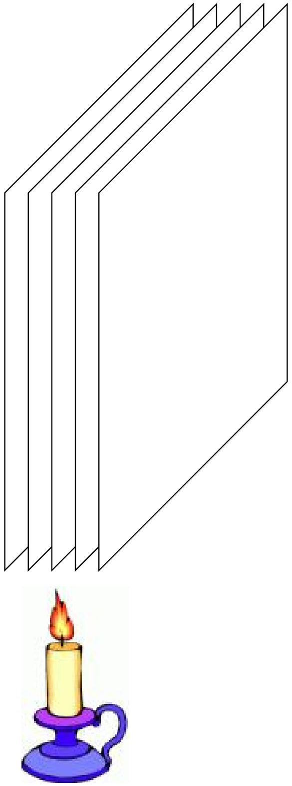
Extensions

Many more examples of gauge/gravity duality have ben constructed including theories with confinement and chiral symmetry breaking
To be able to compute using string theory, the gauge coupling needs to be large at all scales



This is unlike asymptotically free QCD

Finite temperature AdS/CFT correspondence



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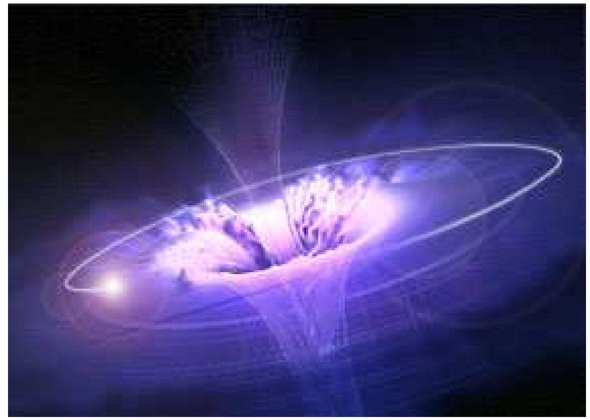


Thermal gauge theory = black hole in anti de-Sitter space

Entropy at strong coupling

Black 3-brane background:

$$ds^2 = \frac{r^2}{R^2}[-f(r)dt^2 + d\vec{x}^2] + \frac{R^2}{r^2 f(r)}dr^2 + R^2 d\Omega_5^2, \qquad f(r) = 1 - \frac{r_0^4}{r^4}$$

Hawking temperature;

$$T_H = \frac{r_0}{\pi R^2}$$

Entropy is computed from area of the horizon, and the result is $S = \pi^6 R^8 T^3 V_{\mathrm{3D}}$. Using AdS/CFT mapping:

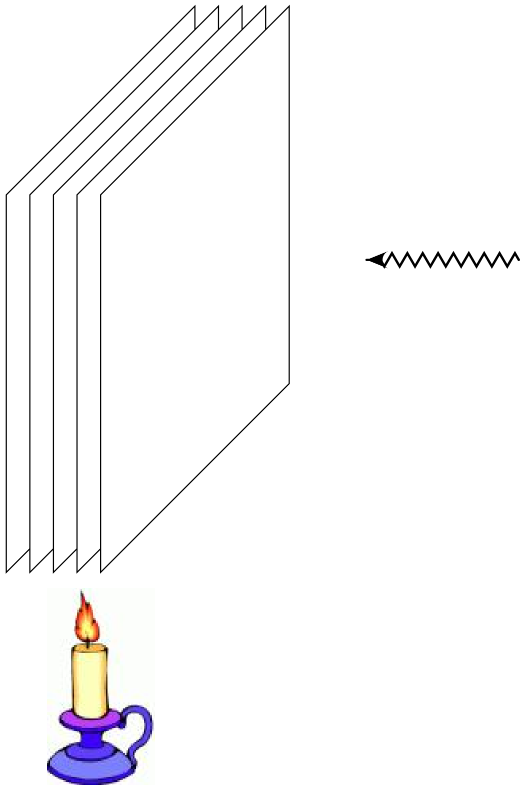
$$s = \frac{S}{V_{\mathrm{3D}}} = \frac{\pi^2}{2} N_c^2 T^3$$

At zero 't Hooft coupling: $s = \frac{2\pi^2}{3} N_c^2 T^3$

$$s(g^2 N_c = \infty) = \frac{3}{4} s(g^2 N_c = 0)$$

Viscosity on the light of duality

Consider a graviton that falls on this stack of N D3-branes
Will be absorbed by the D3 branes.
The process of absorption can be looked at from two different perspectives:



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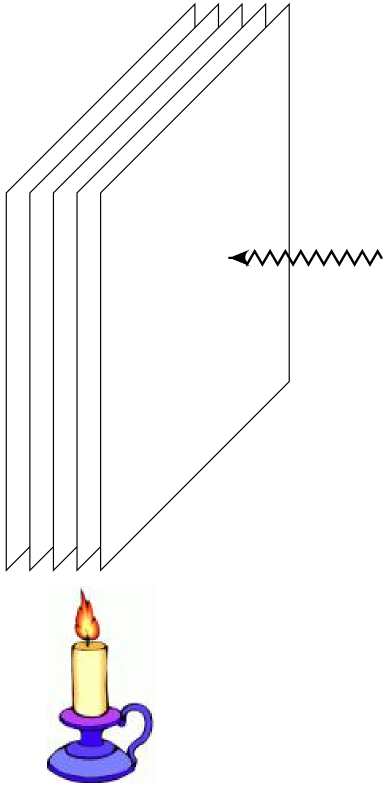

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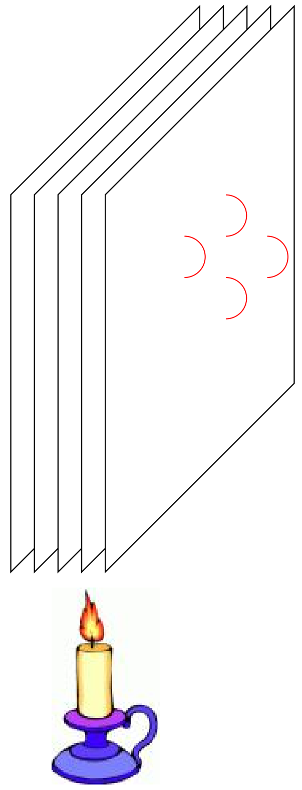

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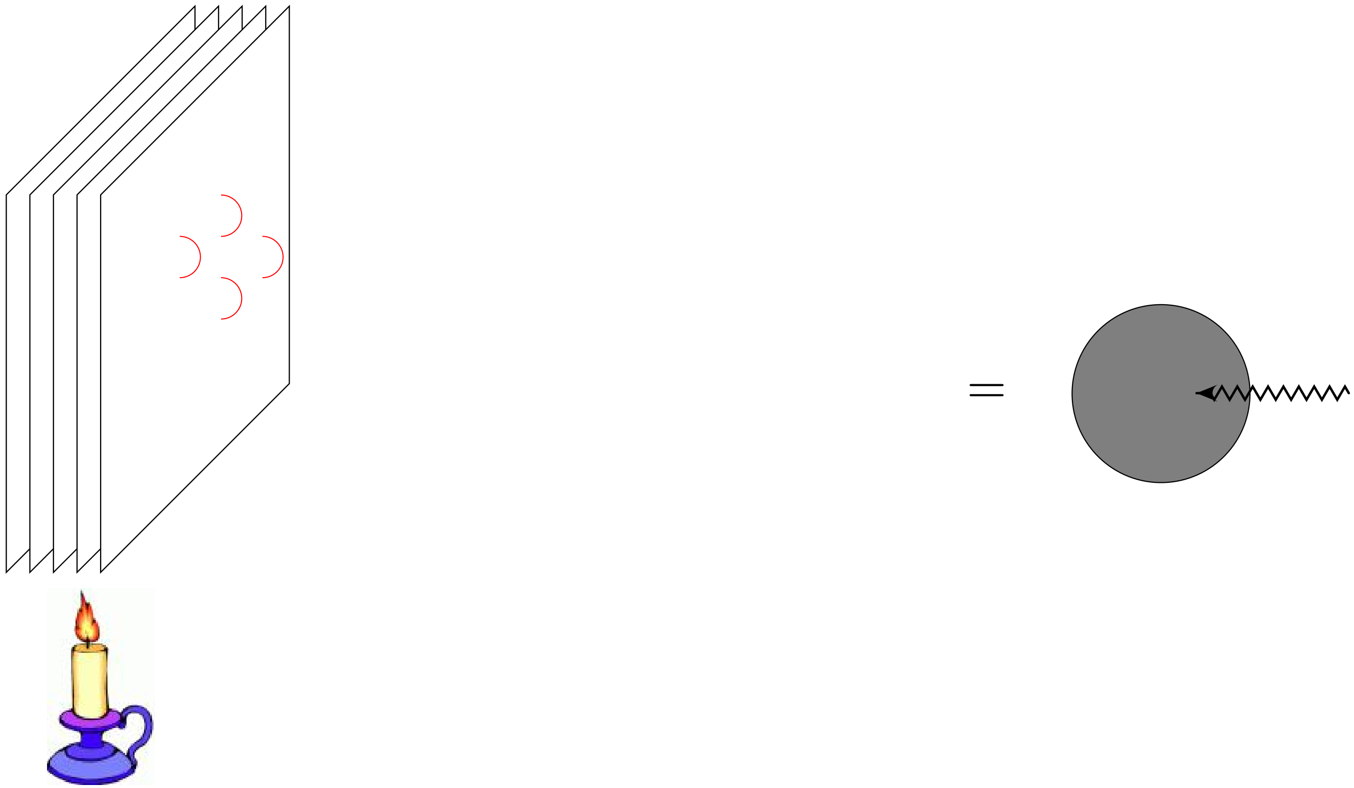

Absorption by D3 branes ($\sim$ viscosity) = absorption by black hole

More formally

- Viscosity is given by Kubo's formula

$$\eta = \lim_{\omega\to 0} \frac{1}{2\omega} \int dt\, d\vec{x}\, e^{i\omega t} \langle [T_{xy}(t,\vec{x}),\, T_{xy}(0,0)] \rangle$$

$$= \lim_{\omega\to 0} \lim_{\vec{q}\to 0} \mathrm{Im}\, G^R_{xy,xy}(\omega,\vec{q})$$

- Via AdS/CFT correspondence, the imaginary part of the retarded Green's function is mapped to the graviton absorption cross section.

$$\sigma_{\mathrm{abs}} = -\frac{16\pi G}{\omega} \mathrm{Im}\, G^R(\omega)$$

- viscosity $\sim$ absorption cross section for low-energy gravitons

$$\eta = \frac{\sigma_{\mathrm{abs}}(0)}{16\pi G}$$

Universality of viscosity/entropy density ratio

- Absorption cross section $=$ area of horizon (follows from a couple of theorems in general relativity)
- Entropy is also proportional to area of horizon: $S = A/(4G)$

$\Rightarrow$ in *all* theories with gravity duals:

$$\frac{\eta}{s} = \frac{\hbar}{4\pi}$$

where η is the shear viscosity, s is the entropy per unit volume.

This is valid in a large, but restricted, class of strongly coupled quantum field theories, which are in a sense infinitely strongly coupled

Boltzmann equation is never used

Viscosity/entropy ratio and uncertainty principle

Estimate of viscosity from kinetic theory

$$\eta \sim \rho v \ell, \qquad s \sim n = \frac{\rho}{m}$$

$$\frac{\eta}{s} \sim m v \ell \sim \hbar \frac{\text{mean free path}}{\text{de Broglie wavelength}}$$

Quasiparticles: de Broglie wavelength $\lesssim$ mean free path

Therefore $\eta/s \gtrsim \hbar$

- Weakly interacting systems have $\eta/s \gg \hbar$.
- Systems with gravity duals are strongly coupled, so it is not suprising that $\eta/s \sim 1$
- What is surprising is *universality*: $\eta/s = \hbar/(4\pi)$ in all theories with gravity duals
- We don't know how to derive the constancy of η/s without AdS/CFT.

Moreover, $\hbar/(4\pi)$ is smaller than η/s achieved in any laboratory liquids: plasmas with gravity duals as the most ideal fluids?

Quark gluon plasma

What is the consequence for the QGP:

- η/s is large both for $T \ll T_c$ (pion gas) and $T \gg T_c$ (weakly coupled QGP)
- minimum near T_c: related to transition from hadrons to quarks? (Csernai, Kapusta, McLerran)
- However: in the weak coupling limit η/s in QCD is larger than η/s in $\mathcal{N} = 4$ SYM at the same 't Hooft coupling by a factor of 7 (Huot, Jeon, Moore)
- So for QGP most likely η/s is above $1/(4\pi)$, in accordance with our hypothesis
- but what is the minimal η/s and how close it is to $1/(4\pi)$ is not clear at this moment
- AdS/CFT suggests η/s as the measure of perfectness of QGP fluid

Conclusion

- Gauge/gravity duality provides unexpected tools to compute the viscosity of some strongly coupled theories
- The class of theories with gravity dual description is limited, but contains very interesting theories with infinite coupling
- The calculation of the viscosity is easy: viscosity $\propto$ absorption cross section of low-energy gravitons by the black hole.
- In this class, the ratio η/s is equal to a universal number $\hbar/4\pi$, much smaller than in any other system in Nature
- The ratio η/s is the measure of perfectness of the QGP

An Introduction to AdS/CFT Correspondence

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Plan of of this talk

- AdS/CFT correspondence at zero temperature.
- Thermodynamics of strongly coupled plasmas ($\mathcal{N} = 4$ supersymmetric Yang-Mills theory)
- Viscosity at strong coupling (if enough time, also my talk at conference)

Not in this talk:

- AdS/CFT inspired models of hadrons
- High-energy scattering
- Jet quenching (Liu's talk at conference)

Motivation

- Strong coupling regime of QCD is difficult
- There exist gauge theories where the strong coupling regime can be studied analytically using AdS/CFT correspondence
- Hope: learn more about QCD from models

Some literature

- Horowitz and Polchinski, Gauge/gravity duality, gr-qc/0602037
  general, philosophical, basic ideas
- Klebanov's TASI lectures Introduction to AdS/CFT correspondence, hep-th/0009139
  genesis of the idea, computation of entropy, correlators
- Gorsky Gauge theories as string theory: the first results, hep-th/0602184
  Wilson loop, viscosity
  but mostly on anomalous dimensions of operators

Zero Temperature AdS/CFT

Original AdS/CFT correspondence

Maldacena; Gubser, Klebanov, Polyakov; Witten

between $N = 4$ supersymmetric Yang-Mills theory
and type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$ds^2 = \frac{R^2}{z^2}(d\vec{x}^2 + dz^2) + R^2 d\Omega_5^2$$

This is a solution to the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

$(T_{\mu\nu} = F_\mu{}^{\alpha\beta\gamma\delta} F_{\nu\alpha\beta\gamma\delta})$

Large 't Hooft limit in gauge theory $\Leftrightarrow$ small curvature limit in string theory

$$g^2 N_c \gg 1 \Leftrightarrow R/l_s = \sqrt{\alpha'}\, R \gg 1$$

Correlation function are computable at large 't Hooft coupling, where string theory $\longrightarrow$ supergravity.

The dictionary of gauge/gravity duality

gauge theory	gravity
operator $\hat{O}$ energy-momentum tensor $T_{\mu\nu}$ dimension of operator	field ϕ graviton $h_{\mu\nu}$ mass of field
globar symmetry conserved current anomaly ...	gauge symmetry gauge field Chern-Simon term ...

$$\int e^{iS_{4\mathrm{D}}+\phi_0 O} = \int e^{iS_{5\mathrm{D}}}$$

where $S_{5\mathrm{D}}$ is computed with nontrivial boundary condition

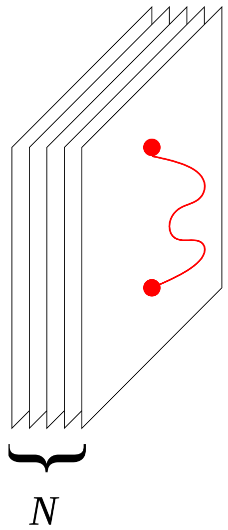
$$\lim_{z\to 0}\phi(\vec{x},z)=\phi_0(\vec{x})$$

Origin of the idea

Consider type IIB string theory in (9+1)D

contains: strings (massless string modes = graviton, dilaton, etc.)
Dp-branes, $p = 1, 3, 5, 7$

Stack N D3 branes on top of each other:



Fluctuations of the branes are described by
$N = 4$ super Yang-Mills theory
field = open strings

$N \gg 1$: space time is curved
closed strings in a curved background

Duality

Hypothesis: the two pictures are two different descriptions of the same object.

Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ $\Leftrightarrow$ $\mathcal{N} = 4$ super Yang-Mills in flat space time a conformal field theory

This is supported by many checks:

- Symmmetries: conformal symmetry $\leftrightarrow$ isometry of AdS_5, R-symmetry $\leftrightarrow$ SO(6) symmmetry of S^5.
- Correlation functions: some can be computed exactly in field theory and checked with AdS/CFT calculations
- ...

Computing correlators

In the limit $N_c \to \infty$, $g^2 N_c \to \infty$, calculation of correlators reduces to solving classical e.o.m:

$$Z[J] = \int D\phi \, e^{iS[\phi] + i\int JO} = e^{iW[J]}$$

$$W[J] = S_{\text{cl}}[\varphi_{\text{cl}}] \text{ : classical action}$$

φ_{cl} solves e.o.m., $\varphi_{\text{cl}}|_{z\to 0} \to J(x)$.

Example: correlator of R-charge currents

- R-charge current in 4D corresponds to gauge field in 5D
- Field equation for transverse components of gauge fields is

$$\partial_\mu(\sqrt{-g}\, g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}) = 0$$

- In the gauge $A_z = 0$, equation for transverse and longitudinal parts of A_μ decouple. The equation for the transverse part is

$$\partial_z \left(\frac{1}{z} \partial_z A_\perp(z,q) \right) - \frac{q^2}{z} A_\perp = 0 \Rightarrow A_\perp(z,q) = QzK_1(Qz) A_\perp(0,q)$$

Computing correlators (continued)

Two-point correlator = second derivative of classical action over boundary values of fields

$$\langle j^\mu j^\nu \rangle \sim \frac{\delta^2 S_{\text{Maxwell}}}{\delta A_\perp(0)^2} = \left(g^{\mu\nu} - \frac{q^\mu q^\nu{}^2}{q} \right) \frac{1}{z} \frac{\partial}{\partial z} [\underbrace{QzK_1(Qz)}_{1+Q^2 z^2 \ln(Qz)}]$$

$$= \#(g^{\mu\nu} q^2 - q^\mu q^\nu) \ln Q^2$$

This structure is the consequence of conformal symmetry

Problem 2

It is known that the operator $O = F_{\mu\nu}^2$ in 4D field theory corresponds to the dilaton field ϕ in 5D. The dilaton couples minimally to the metric:

$$S = -\frac{1}{2}\int d^5x\,\sqrt{-g}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

- Find the general solution to the equation of motion for ϕ in AdS space

$$ds^2 = \frac{R^2}{z^2}(dz^2 + d\vec{x}^2)$$

  with the boundary condition $\phi(z=0,\vec{x}) = \phi_0(\vec{x})$, $\phi(z=\infty,\vec{x}) = 0$.
- Find the classical action S as a functional of $\phi_0(\vec{x})$,
- Find the correlation function of O,

$$\langle O(\vec{x})O(\vec{y})\rangle = \frac{\delta^2 S}{\delta\phi(\vec{x})\delta\phi(\vec{y})}$$

Ref.: Gubser, Klebanov, Polyakov 1998

Plasma Thermodynamics and Black Holes

Reminder of GR and black holes

Metric: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$

Einstein equation: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$

where $R_{\mu\nu}$ is the Ricci tensor ($\sim$ curvature, $\partial^2 g_{\mu\nu}$)
$T_{\mu\nu}$ is the stress-energy tensor of matter

Example: Schwarzschild black hole is a solution with $T_{\mu\nu} = 0$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \underbrace{(d\theta^2 + \sin^2\theta d\phi^2)}_{d\Omega_2^2}$$

Properties

- $r \to \infty$: flat space
- $r = r_0 = 2GM$: metric is singular, but curvature is finite (coordinate singularity)
  this is the black hole horizon

Behavior near horizon

Near $r = r_0$, $ds^2 = \#(r - r_0)dt^2 + \#dr^2/(r - r_0) + \cdots$

For r near r_0, we introduce new coordinate ρ:

$$r - r_0 = \frac{\rho^2}{4r_0} \Longrightarrow ds^2 = -\frac{\rho^2}{4r_0^2}dt^2 + d\rho^2 + r_0^2 d\Omega_2^2$$

This is simply a Minkowski version of polar coordinates:

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2, \qquad \varphi = \frac{it}{2r_0^2}$$

and has no curvature singularity at $\rho = 0$, if φ is a periodic variable

<u>Hawking temperature</u>: $\varphi \sim \varphi + 2\pi$

corresponding to periodic Euclidean time: $it \sim it + \underbrace{4\pi r_0}_{\beta = 1/T}$

$$T_H = \frac{1}{4\pi r_0}$$

Black hole entropy

We inteprete the black hole as a thermodynamical system with energy

$$E = M = \frac{r_0}{2G}$$

and temperture $T = T_H$.

$$dS = \frac{dE}{T} = 4\pi r_0 \frac{dr_0}{2G}$$

$$S = \frac{\pi r_0^2}{G} = \frac{A}{4G}$$

$A = 4\pi r_0^2$ area of horizon

Finite-temperature AdS/CFT correspondence

Black 3-brane solution:

$$ds^2 = \frac{r^2}{R^2}[-f(r)dt^2 + d\vec{x}^2] + \frac{R^2}{r^2 f(r)}dr^2 + R^2 d\Omega_5^2, \qquad f(r) = 1 - \frac{r_0^4}{r^4}$$

- $r_0 = 0$, $f(r) = 1$: is AdS$_5\times$ S^5, $r = R^2/z$.
- $r_0 \neq 0$: corresponds to $\mathcal{N} = 4$ SYM at temperture

$$T = T_H = \frac{r_0}{\pi R^2}$$

Problem 3

- Check that the 5D part of the metric,

$$ds^2 = \frac{r^2}{R^2}[-f(r)dt^2 + d\vec{x}^2] + \frac{R^2}{r^2 f(r)}dr^2, \qquad f(r) = 1 - \frac{r_0^4}{r^4}$$

  satisfies the equation

$$R_{\mu\nu} - g_{\mu\nu}R = \Lambda g_{\mu\nu}$$

  where Λ is a negative cosmological constant. You might want to use a symbolic manipulation program (e.g., Mathematica) and a a package to compute the curvature tensor (e.g., GRTensor, Ricci).

- Check the formula for the Hawking temperature.

Entropy density

Entropy $= A/4G$

A is the area of the event horizon
G is the 10D Newton constant.

$$A = \int dx\, dy\, dz\, \sqrt{g_{xx} g_{yy} g_{zz}} \times \underbrace{\pi^3 R^5}_{\text{area of S}^5} = V_{3D} \pi^3 r_0^3 R^2 = \pi^6 V_{3D} R^8 T^3$$

On the other hand, from AdS/CFT dictionary

$$R^4 = \frac{\sqrt{8\pi G}}{2\pi^{5/2}} N_c$$

(for derivation see e.g., Klebanov hep-th/0009139).
Therefore

$$S = \frac{\pi^2}{2} N_c^2 T^3 V_{\text{3D}}$$

This formula has the same N^2 behavior as at zero 't Hooft coupling $g^2 N_c = 0$ but the numerical coefficient is 3/4 times smaller.

Thermodynamics

$$S = f(g^2 N_c) N_c^2 T^3 V_{3D}$$

where the function f interpolates between weak-coupling and strong-coupling values, which differ by a factor of 3/4.

Problem 4

The $\mathcal{N}=4$ SYM theory contains: one gauge boson, 4 Weyl fermions and 6 real scalar fields. Eachh field is in the adjoint representation of the gauge group SU(N_c). Find the entropy density at zero 't Hooft coupling and finite temperature, show that it is 4/3 times larger than the value found from AdS/CFT correspondence at infinite 't Hooft coupling.

Hydrodynamics from Black Hole Physics

Hydrodynamics

- is the effective theory describing the long-distance, low-frequency behavior of interacting finite-temperature systems. Hydrodynamic regime
- Valid at distances $\gg$ mean free path, time $\gg$ mean free time.
- At these length/time scales: local thermal equilibrium: T, μ vary slowly in space.
- Simplest example of a hydrodynamic theory: the Navier-Stokes equations

$$\partial_t \rho + \nabla \dot{} (\rho \mathbf{v}) = 0$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \eta \nabla^2 \mathbf{v}$$

- The quark-gluon plasma can be described by similar equations.
- All microscopic physics reduces to a small number of *kinetic coefficients* (shear viscosity η, bulk viscosity, diffussion coeffecients).

Idea

The idea is to use AdS/CFT correspondence to explore the hydrodynamic regime of thermal gauge theory.

- Finite-T QFT $\Leftrightarrow$ black hole with translationally invariant horizon

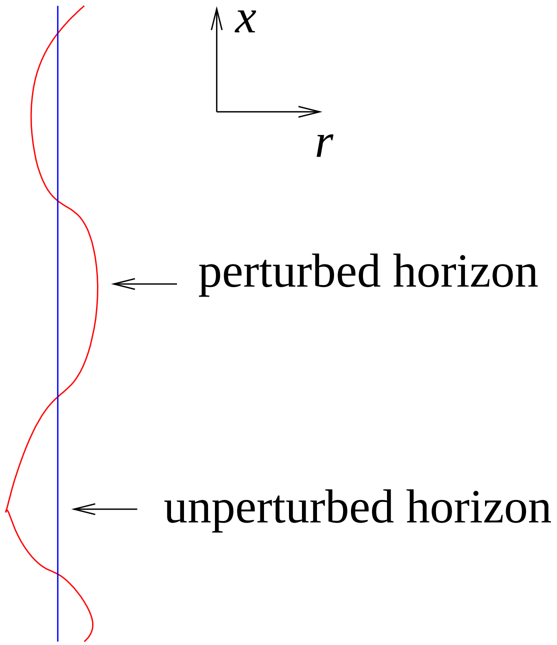
$$ds^2 = \frac{r^2}{R^2}(-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}(\frac{dr^2}{f} + r^2 \partial\Omega_5^2)$$

  horizon: $r = r_0$, $\vec{x}$ arbitrary.

- Local thermal equilibrium $\Leftrightarrow$ parameters of metric (e.g., r_0) slowly vary with $\vec{x}$.

  remember that $T \sim r_0/R^2$.

Dynamics of the horizon



$$T \sim r_0 = r_0(\vec{x})$$

Generalizing black hole thermodynamics M, Q,... to black brane hydrodynamics
$T = T_H(\vec{x})$, $\mu = \mu(\vec{x})$

While we know $S = A/(4G)$, what is the viscosity from the point of view of gravity?

Kubo's formula

$$\eta = \lim_{\omega\to 0}\frac{1}{2\omega}\int dt\, d\vec{x}\, e^{i\omega t}\langle[T_{xy}(t,\vec{x}),\, T_{xy}(0,0)]\rangle$$
$$= \lim_{\omega\to 0}\lim_{\vec{q}\to 0}\mathrm{Im}\, G^R_{xy,xy}(\omega,\vec{q})$$

imaginary part of the retarded correlator of T_{xy}.

Similar relations exist for other kinetic coefficients.

Gravity counterpart of Kubo's formula

Klebanov: $\mathrm{Im} G^R$ is proportional to the absorption cross section by the black hole.

$$\sigma_{\mathrm{abs}} = -\frac{16\pi G}{\omega} \mathrm{Im}\, G^R(\omega)$$

That means viscosity = absorption cross section for low-energy gravitons

$$\eta = \frac{\sigma_{\mathrm{abs}}(0)}{16\pi G}$$

The absorption cross section can be found classically.
There is a theorem that the cross section at $\omega = 0$ is equal to the area of the horizon.
But the entropy is also proportional to the area of the horizon

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Viscosity/entropy ratio and uncertainty principle

Estimate of viscosity from kinetic theory

$$\eta \sim \rho v \ell, \qquad s \sim n = \frac{\rho}{m}$$

$$\frac{\eta}{s} \sim m v \ell \sim \hbar \frac{\text{mean free path}}{\text{de Broglie wavelength}}$$

Quasiparticles: de Broglie wavelength $\lesssim$ mean free path

Therefore $\eta/s \gtrsim \hbar$

- Weakly interacting systems have $\eta/s \gg \hbar$.
- Theories with gravity duals have universal η/s, but we don't know how to derive the constancy of η/s without AdS/CFT.

Further references

- Viscosity from AdS/CFT correspondence: Policastro, DTS, Starinets; Kovtun, DTS, Starinets
- Energy loss of a heavy quark: Herzog et al 2006
- $\hat{q}$ parameter: Liu, Rajagopal, Wiedemann 2006.