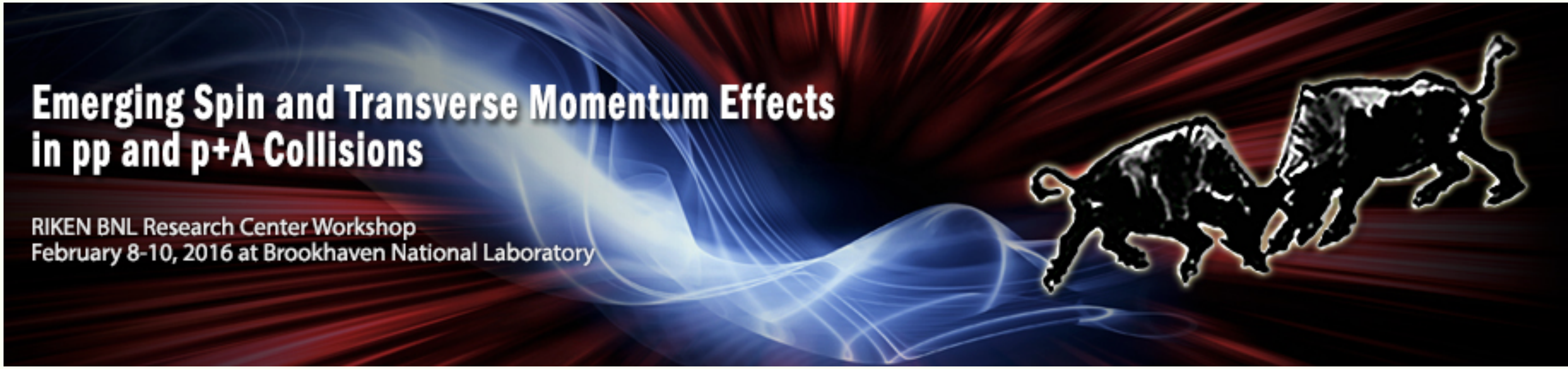
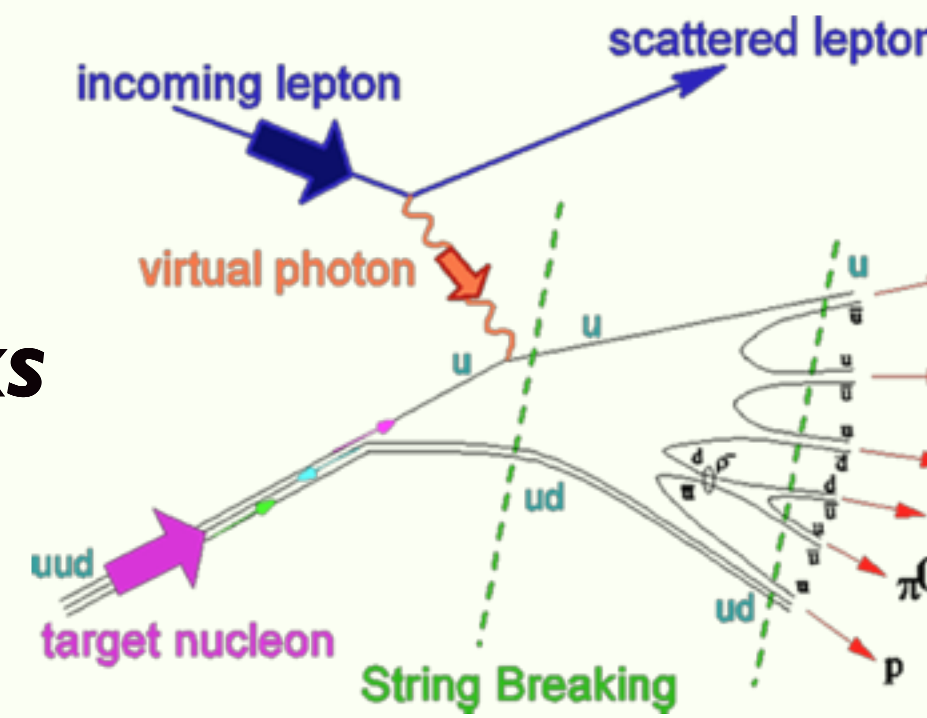
# *Bessel Weighting*





***Leonard Gamberg Penn State Berks***


Boer, LG, Musch,Prokudin JHEP 2011

M. Aghasyan, H. Avakian, E. De Sanctis, LG, M. Mirazita, B. Musch, A. Prokudin, P. Rossi JHEP 2015

# Outline

- BWA in Parton model connection w/ conventional weighting
- Impact on studying BW and TMD evolution
- Sketch ... Elements TMD Factorization-SIDIS
- Cancellation of Universal & flavor indep. factors in BWAs
- A study of BW of experimental observables $A_{LL}(b_T)$

# Comments on Weighting

- Weighting enables one disentangle in a model independent way the CS in terms of transverse momentum moments of TMDs

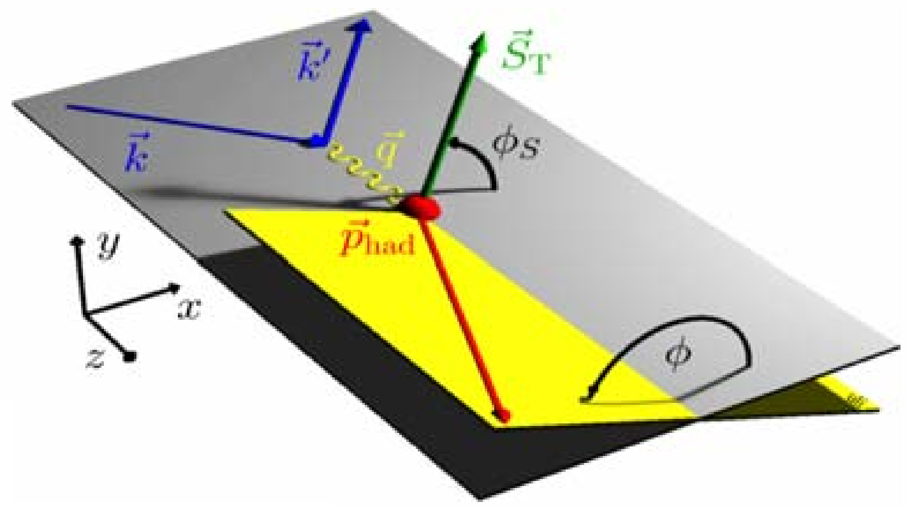
- Convert convolutions in the cross section into simple products
  not a new idea **Kotzinian, Mulders PLB 97, Boer, Mulders PRD 98**

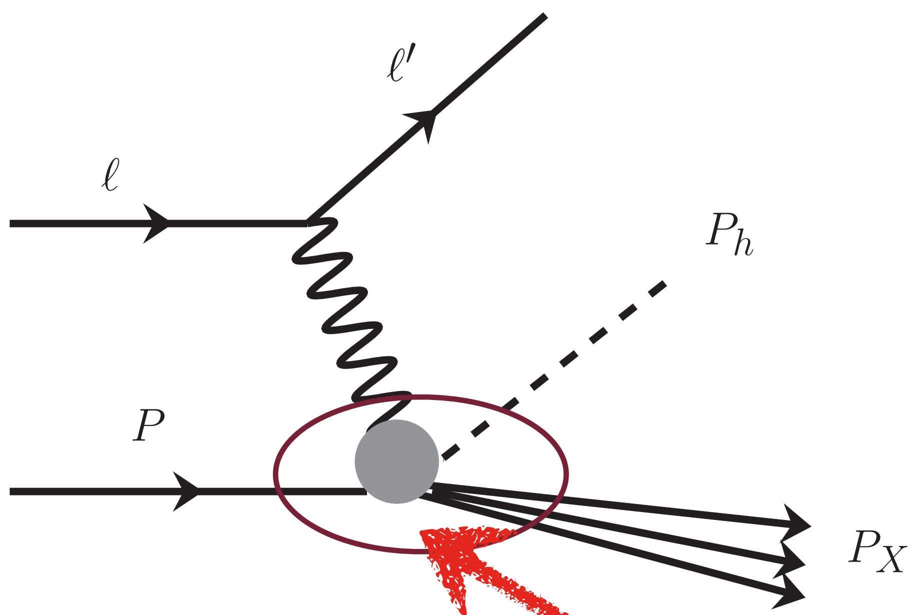
- Bessel Weighting solves problem of infinite contribution from large transverse momentum that arise from using "conventional weighting
  **Boer, Gamberg,Musch,Prokudin JHEP 2011**

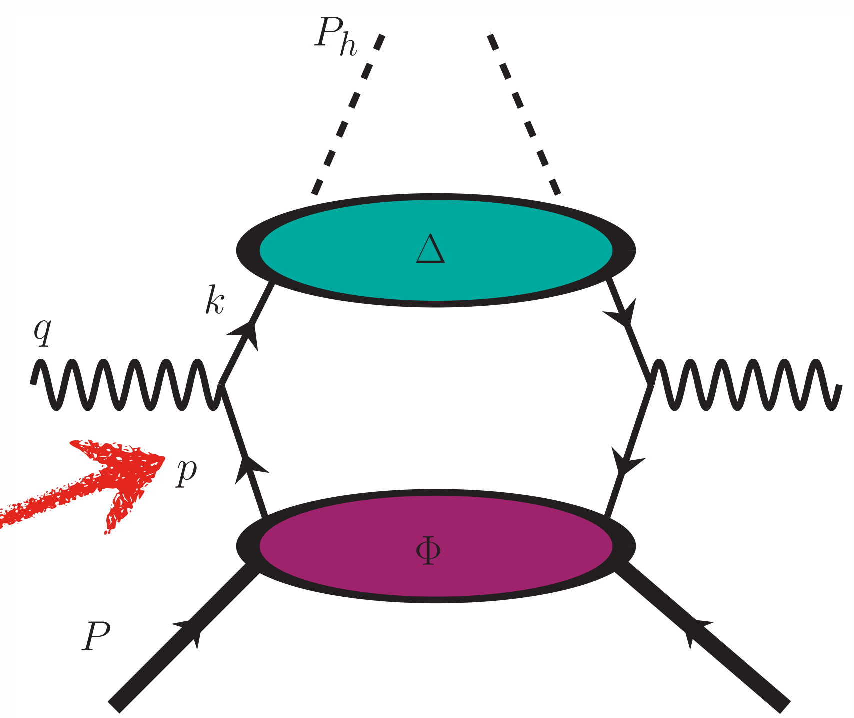
- Explore impact these BWA have on studying the scale dependence of the SIDIS cross section at small to moderate transverse momentum where the TMD framework is expected to give a good description of the cross section **Boer, Gamberg, Musch,Prokudin JHEP 2011**

# Factorization  Parton Model SIDIS

*Kotzinian NPB 95,*
*Mulders Tangermann NPB 96,*
*Boer & Mulders PRD 97*
*Bacchetta et al JHEP 08*




1photon exchange
approx Factorize
Hadronic Tensor

$$\frac{d\sigma}{dx_B\, dy\, d\psi\, dz_h\, d\phi_h\, |\boldsymbol{P}_{h\perp}|\, d|\boldsymbol{P}_{h\perp}|} = \frac{\alpha^2}{x_B y Q^2} \frac{y^2}{(1-\varepsilon)} \left(1 + \frac{\gamma^2}{2x_B}\right) L_{\mu\nu} W^{\mu\nu},$$

# Factorization $P_T$ of hadron small sensitive to intrinsic transv. momentum of partons

$$W^{\mu\nu}(q,P,S,P_h) = \int \frac{d^2\mathbf{p}_T}{(2\pi)^2}\int \frac{d^2\mathbf{k}_T}{(2\pi)^2}\delta^2\left(\mathbf{p}_T - \frac{\mathbf{P}_{h\perp}}{z_h} - \mathbf{k}_T\right)\mathrm{Tr}\left[\Phi(x,\mathbf{p}_T)\gamma^\mu\Delta(z,\mathbf{k}_T)\gamma^\nu\right]$$

$$\Phi(x,\mathbf{p}_T) = \int dp^- \Phi(p,P,S)|_{p^+=x_BP^+}\ , \qquad \Delta(z,\mathbf{k}_T) = \int dk^- \Delta(k,P_h)|_{k^-=\frac{P^-}{z_h}}$$

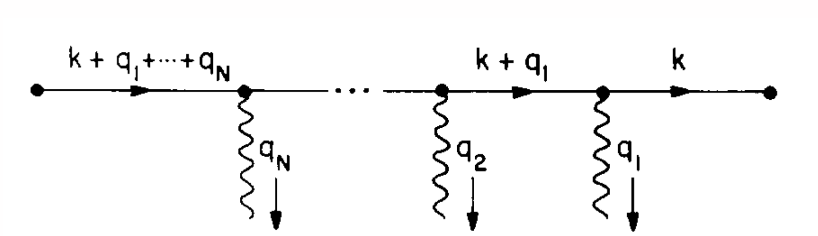
*Small transverse momentum*

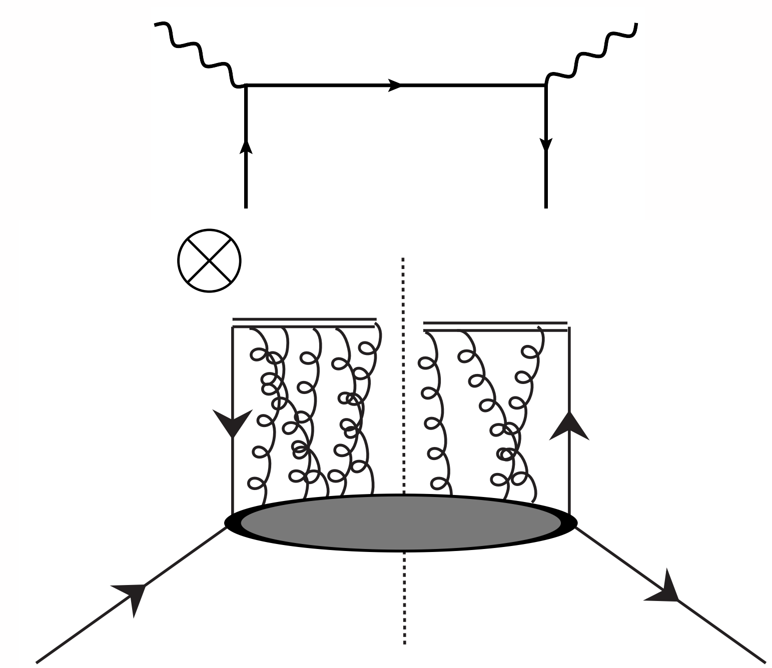
<u>Purely Kinematic-integrate over small momentum component</u>

“Gauge invariant extension” of parton model

Collins & Soper NPB193 (81) & Efremov, Radyushkin Theo. Math. Phys. 81... also Collins Found. PQCD 2011

respect gauge invariance **color** gauge invariance



# Partonic picture Structure Functions momentum CONVOLUTION

$$\mathcal{C}\left[w\, f\, D\right] = x \sum_a e_a^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^{(2)}\left(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z\right) w(\boldsymbol{p}_T, \boldsymbol{k}_T)\, f^a(x, p_T^2)\, D^a(z, k_T^2)$$

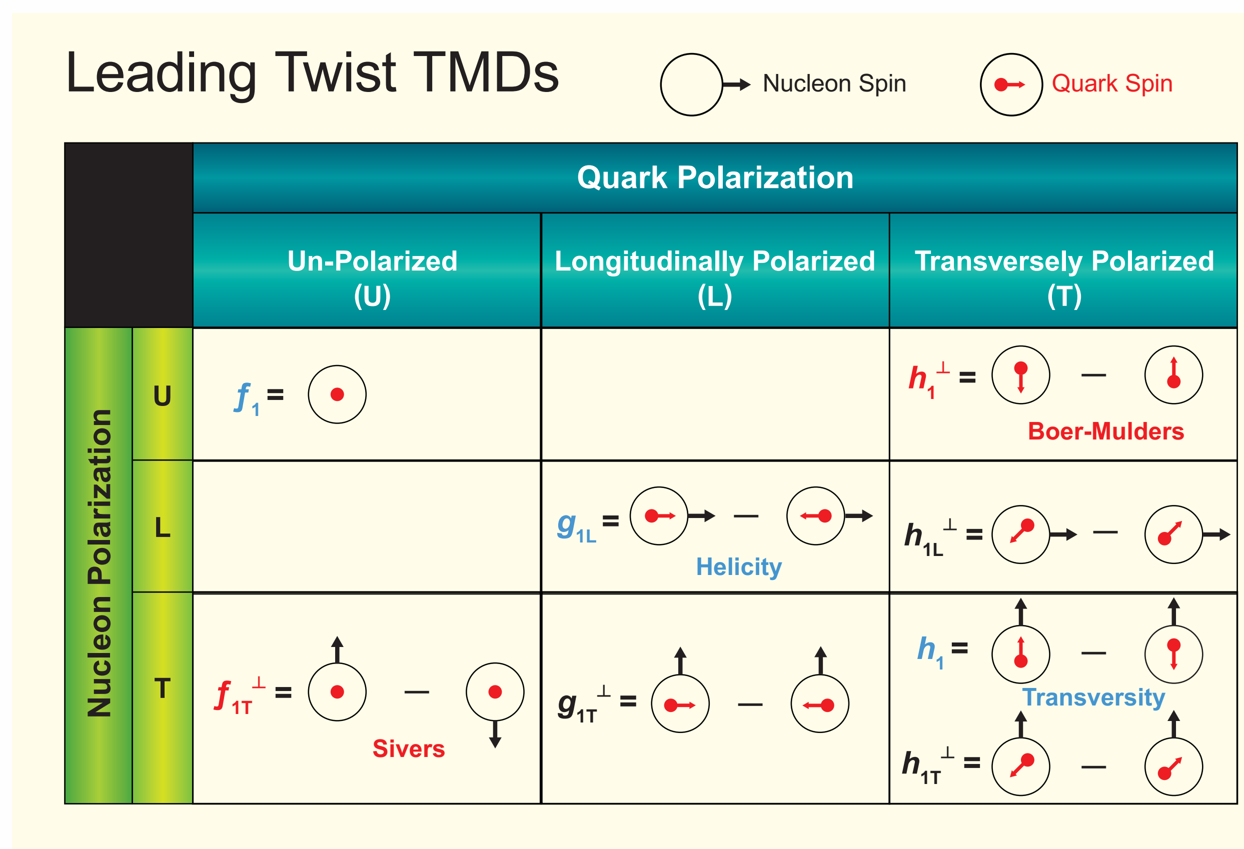
$$F_{UU,T} = \mathcal{C}\left[f_1 D_1\right],$$

$$F_{LL} = \mathcal{C}\left[g_{1L} D_1\right],$$

$$F_{UT,T}^{\sin(\phi_h - \phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}} \cdot \boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\right],$$

$$F_{UT}^{\sin(\phi_h + \phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}} \cdot \boldsymbol{k}_T}{M_h} h_1 H_1^{\perp}\right],$$

Sivers PRD 1990, Brodsky Hwang Schmidt 2002 PLB, Collins PLB 2002



# SIDIS CS & leading and subleading twist structure functions

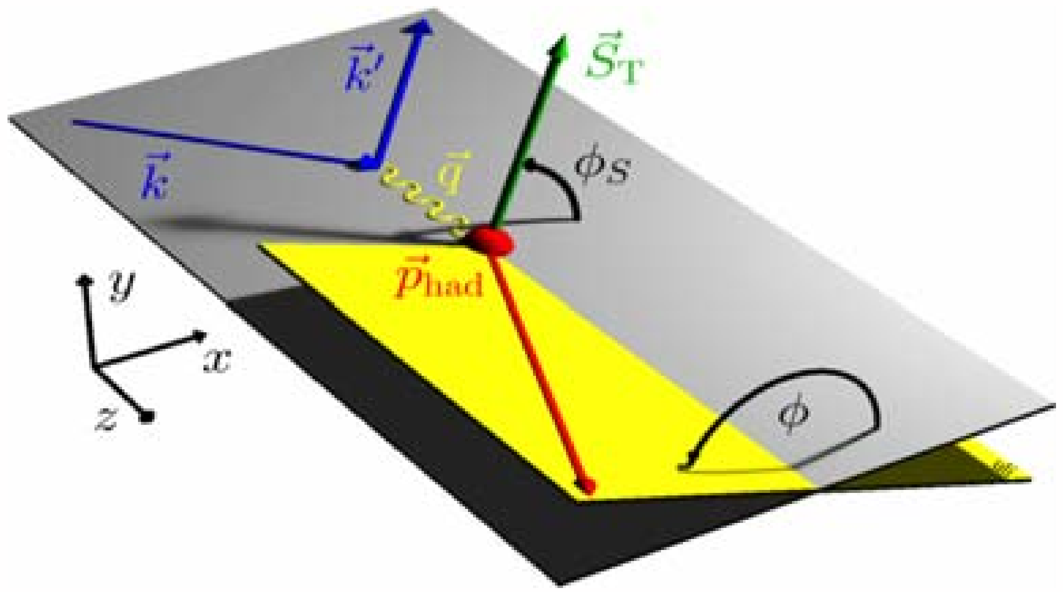
$$
\begin{aligned}
\frac{d\sigma}{dx_B\, dy\, d\psi\, dz_h\, d\phi_h\, dP_{h\perp}^2} &= \frac{\alpha^2}{x_B y Q^2} \frac{y^2}{2\,(1-\varepsilon)} \left(1+\frac{\gamma^2}{2x_B}\right) \Bigg\{ F_{UU,T} + \varepsilon F_{UU,L} + \sqrt{2\,\varepsilon(1+\varepsilon)}\, \cos\phi_h\, F_{UU}^{\cos\phi_h} \\
&+ \quad \varepsilon \cos(2\phi_h)\, F_{UU}^{\cos 2\phi_h} + \lambda_e\, \sqrt{2\,\varepsilon(1-\varepsilon)}\, \sin\phi_h\, F_{LU}^{\sin\phi_h} \\
&+ \quad S_\parallel \left[ \sqrt{2\,\varepsilon(1+\varepsilon)}\, \sin\phi_h\, F_{UL}^{\sin\phi_h} + \varepsilon \sin(2\phi_h)\, F_{UL}^{\sin 2\phi_h} \right] \\
&+ \quad S_\parallel \lambda_e \left[ \sqrt{1-\varepsilon^2}\, F_{LL} + \sqrt{2\,\varepsilon(1-\varepsilon)}\, \cos\phi_h\, F_{LL}^{\cos\phi_h} \right] \\
&+ \quad |\boldsymbol{S}_\perp| \Bigg[ \sin(\phi_h-\phi_S) \left( F_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon\, F_{UT,L}^{\sin(\phi_h-\phi_S)} \right) \\
&+ \quad \varepsilon\, \sin(\phi_h+\phi_S)\, F_{UT}^{\sin(\phi_h+\phi_S)} + \varepsilon\, \sin(3\phi_h-\phi_S)\, F_{UT}^{\sin(3\phi_h-\phi_S)} \\
&+ \quad \sqrt{2\,\varepsilon(1+\varepsilon)}\, \sin\phi_S\, F_{UT}^{\sin\phi_S} + \sqrt{2\,\varepsilon(1+\varepsilon)}\, \sin(2\phi_h-\phi_S)\, F_{UT}^{\sin(2\phi_h-\phi_S)} \Bigg] \\
&+ \quad |\boldsymbol{S}_\perp| \lambda_e \Bigg[ \sqrt{1-\varepsilon^2}\, \cos(\phi_h-\phi_S)\, F_{LT}^{\cos(\phi_h-\phi_S)} + \sqrt{2\,\varepsilon(1-\varepsilon)}\, \cos\phi_S\, F_{LT}^{\cos\phi_S} \\
&+ \quad \sqrt{2\,\varepsilon(1-\varepsilon)}\, \cos(2\phi_h-\phi_S)\, F_{LT}^{\cos(2\phi_h-\phi_S)} \Bigg] \Bigg\},
\end{aligned}
$$

*Bacchetta et al JHEP 08*

# Observables SIDIS-CS expressed structure functions

$$\frac{d^6\sigma}{dxdydzd\phi_h dP_{h\perp}^2} \sim \left\{ F_{UU,T} \cdots + \ldots |S_\perp| \left( \sin(\phi_h - \phi_S) F_{UT,T}^{\sin(\phi_h-\phi_S)} + \sin(\phi_h + \phi_S) \varepsilon F_{UT}^{\sin(\phi_h+\phi_S)} \ldots \right) \ldots \right\}$$

*Kotzinian NPB 95,*
*Mulders Tangermann NPB 96,*
*Boer & Mulders PRD 97*
*Bacchetta et al JHEP 08*




## Spin asymmetry projected $\mathcal{P}$ from cross section

$$\mathcal{A}_{XY}^{\mathcal{P}} \equiv \frac{\int d\phi_h d\phi_S \mathcal{P}(\phi_h, \phi_S) \left(d\sigma^\uparrow - d\sigma^\downarrow\right)}{\int d\phi_h d\phi_S \left(d\sigma^\uparrow + d\sigma^\downarrow\right)}$$

$XY$-polarization e.g.

$$\mathcal{P}(\phi_h, \phi_S) = \sin(\phi_h - \phi_s)$$

# Weighted asymmetries proposed as *model independent deconvolution* of CS in terms of moments of TMDs

**Kotzinian, Mulders PLB 97, Boer, Mulders PRD 98**

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$$\text{e.g. } w_1(\boldsymbol{P}_{h\perp}) = \frac{|\boldsymbol{P}_{h\perp}|}{zM}$$

$$A_{UT,T}^{w_1 \sin(\phi_h - \phi_S)} = 2\,\frac{\int d|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{P}_{h\perp}| d\phi_h\, d\phi_S\, w_1(|\boldsymbol{P}_{h\perp}|)\ \sin(\phi_h - \phi_S)\ \{d\sigma(\phi_h, \phi_S) - d\sigma(\phi_h, \phi_S + \pi)\}}{\int d|\boldsymbol{P}_{h\perp}|\, d\phi_h\, |\boldsymbol{P}_{h\perp}| d\phi_S\, w_0(|\boldsymbol{P}_{h\perp}|)\ \{d\sigma(\phi_h, \phi_S) + d\sigma(\phi_h, \phi_S + \pi)\}},$$

$$A_{UT}^{\frac{|\boldsymbol{P}_{h\perp}|}{z_h M} \sin(\phi_h - \phi_s)} = -2\,\frac{\sum_a e_a^2\ f_{1T}^{\perp(1)}(x)\ D_1^{a(0)}(z)}{\sum_a e_a^2\ f_1^{a(0)}(x)\ D_1^{a(0)}(z)}$$

**Undefined w/o subtractions prescription-need regularization to subtract infinite contribution at large transverse momentum**

Models studies ...
Gamberg, Golstein, Oganesyan PRD 2003
Conti Bacchetta Radici Eur.Phys.J. 2010

# Problem with $k_T$ moments

$$f_{1T}^{\perp(1)}(x) = \int d^2k_T \frac{k_T^2}{2M} f_{1T}^{\perp}(x, k_T)$$

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$$f_{1T}^{\perp(1)}(x) = \int d^2k_T \frac{k_T^2}{2M} f_{1T}^{\perp}(x, k_T)$$

$$f_{1T}^{\perp}(x, k_T) \sim \frac{M^2}{\left(k_T^2 + M^2\right)^2}$$

- power counting ... Sivers tail

  Bacchetta et al. JHEP 08, Aybat, Collins, Rogers,Qiu PRD 2012

- Moment diverges

# “Now for something completely different”

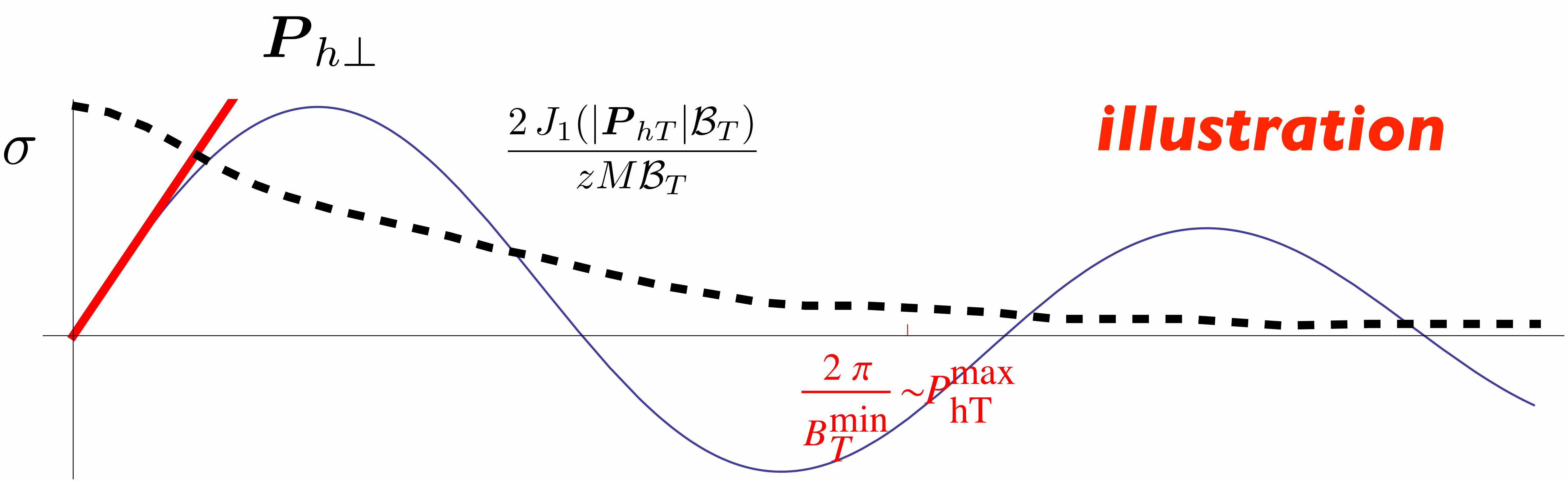
- Change the $w_1(\boldsymbol{P}_{h\perp}) = \frac{|\boldsymbol{P}_{h\perp}|}{zM}$ weight to a *Bessel function*

$$\frac{2\,J_1(|\boldsymbol{P}_{hT}|\mathcal{B}_T)}{zM\mathcal{B}_T}$$

- why on earth would you do that?!

# More sensitive to low $P_{h\perp}$ region

$\mathcal{B}_T$ can serve as a lever arm to enhance the low $P_{h\perp}$ description and possibly dampen *lg.* momentum tail of moments and cross section. For this need investigate the full TMD factorization formalsim in b-space




More formally this picture emerges from formalism on scale dependence of TMDs & TMD evolution

# Comments on Weighting

- Bessel weighting is a natural outgrowth of re-writing SIDIS cross section (or DY or $e^+e^-$ ) in coordinate space
- *nb* the solution to this problem is to consider TMD evolution in "*b*-coordinate space". Seed of idea is in CSS work of 1981/1982 see John's Talk today on CSS and JCC formalism & Ignazio's approach ...

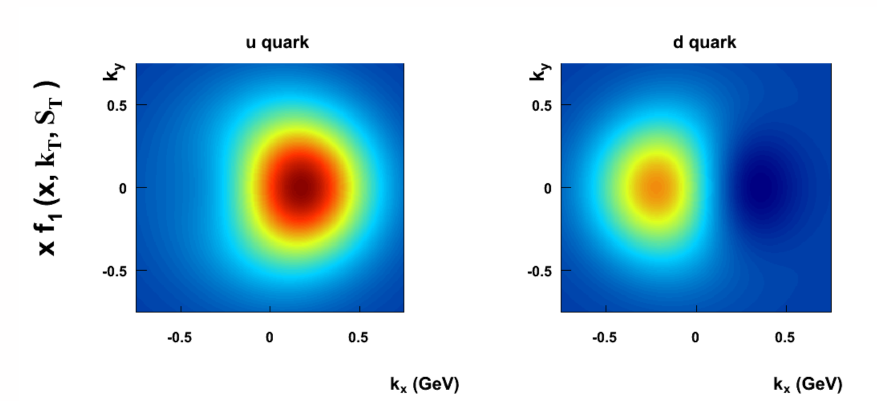
# e.g. BW Example Sivers Function "parton model"

## Fourier Transform Convolution of Structure function

$$\mathcal{C}\big[w\, f\, D\big] = x \sum_a e_a^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^{(2)}\big(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z\big)\, w(\boldsymbol{p}_T, \boldsymbol{k}_T)\, f^a(x, p_T^2)\, D^a(z, k_T^2)$$

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\right],$$

<u>"dipole structure"</u>




$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = -x_B \sum_a e_a^2 \int \frac{d|\boldsymbol{b}_T|}{(2\pi)} |\boldsymbol{b}_T|^2 J_1(|\boldsymbol{b}_T|\,|\boldsymbol{P}_{h\perp}|) M z\; \tilde{f}_{1T}^{\perp a(1)}(x, z^2\boldsymbol{b}_T^2)\, \tilde{D}_1^a(z, \boldsymbol{b}_T^2).$$

$\tilde{f}_1$, $\tilde{f}_{1T}^{\perp(1)}$, and $\tilde{D}_1$ are Fourier Transf. of TMDs/FFs
convolution in momentum becomes product in *b*-space

Boer, LG, Musch, Prokudin JHEP 2011

# Pretzelocity and Collins

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = \mathcal{C}\left[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)\,(\boldsymbol{p}_T\cdot\boldsymbol{k}_T) + \boldsymbol{p}_T^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T) - 4\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)}{2M^2M_h}\,h_{1T}^{\perp}H_1^{\perp}\right]$$

Write in "cylindrical polar"- is traceless irreducible tensor no mixture of Bessel "$J_3$"

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = x_B\sum_a e_a^2\int\frac{d|\boldsymbol{b}_T|}{(2\pi)}|\boldsymbol{b}_T|^4\,J_3(|\boldsymbol{b}_T|\,|\boldsymbol{P}_{h\perp}|)\,\frac{M^2M_hz^3}{4}\,\tilde{h}_{1T}^{\perp a(2)}(x,z^2\boldsymbol{b}_T^2)\,\tilde{H}_1^{\perp a(1)}(z,\boldsymbol{b}_T^2)\,.$$

Simple product "$\mathcal{P}$ "

# Comments on Weighting

*The FT transform of the e.g. Sivers asympt. reduces to first moment of Sivers TMD*

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T) \equiv \frac{2}{M^2} \frac{\partial}{\partial b_T^2} \tilde{f}_{1T}^{\perp}(x, b_T)$$

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T) = \frac{2\pi}{M^2} \int_0^\infty dk_T \, \frac{k_T^2}{b_T} \, J_1(k_T \, b_T) \; f_{1T}^{\perp}(x, k_T)$$

$$\lim_{b_T \to 0} \tilde{f}_{1T}^{\perp(1)}(x, b_T) = \frac{2}{M^2} 2\pi \int_0^\infty dk_T \, \frac{k_T^2}{2b_T} \, \frac{k_T \, b_T}{2} f_{1T}^{\perp}(x, k_T)$$

$$\lim_{b_T \to 0} \tilde{f}_{1T}^{\perp(1)}(x, 0) = f_{1T}^{\perp(1)}(x)$$

## ★ CS has simpler *S/T* interpretation--multipole expansion in terms of $b_T\,[\mathrm{GeV}^{-1}]$ conjugate to $\boldsymbol{P}_{h\perp}$

$$
\begin{aligned}
&\frac{d\sigma}{dx_B\, dy\, d\phi_S\, dz_h\, d\phi_h\, |\boldsymbol{P}_{h\perp}| d|\boldsymbol{P}_{h\perp}|} = \\
&\frac{\alpha^2}{x_B y Q^2}\,\frac{y^2}{(1-\varepsilon)}\left(1+\frac{\gamma^2}{2x_B}\right)\int \frac{d|\boldsymbol{b}_T|}{(2\pi)}|\boldsymbol{b}_T|\Big\{ J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,T} + \varepsilon\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,L} \\
&+ \sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos\phi_h} + \varepsilon\cos(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos(2\phi_h)} \\
&+ \lambda_e\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LU}^{\sin\phi_h} \\
&+ S_\parallel\Big[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin\phi_h} + \varepsilon\sin(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin 2\phi_h}\Big] \\
&+ S_\parallel\lambda_e\Big[\sqrt{1-\varepsilon^2}\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL} + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL}^{\cos\phi_h}\Big] \\
&+ |\boldsymbol{S}_\perp|\Big[\sin(\phi_h-\phi_S)\, \underline{J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)}\,\Big(\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon\,\mathcal{F}_{UT,L}^{\sin(\phi_h-\phi_S)}\Big) \\
&\quad + \varepsilon\,\sin(\phi_h+\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} \\
&\quad + \varepsilon\,\sin(3\phi_h-\phi_S)\, \underline{J_3(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)}\,\mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)} \\
&\quad + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_S\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin\phi_S} \\
&\quad + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin(2\phi_h-\phi_S)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(2\phi_h-\phi_S)}\Big] \\
&+ |\boldsymbol{S}_\perp|\lambda_e\Big[\sqrt{1-\varepsilon^2}\,\cos(\phi_h-\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LT}^{\cos(\phi_h-\phi_S)} \\
&\quad + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LT}^{\cos\phi_S} \\
&\quad + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos(2\phi_h-\phi_S)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LT}^{\cos(2\phi_h-\phi_S)}\Big]\Big\}
\end{aligned}
$$

Sivers

# FT Structure Functions

$$
\begin{aligned}
\mathcal{F}_{UU,T} &= \mathcal{P}[\tilde{f}_1^{(0)}\ \tilde{D}_1^{(0)}]\,, \\
\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} &= -\mathcal{P}[\tilde{f}_{1T}^{\perp(1)}\ \tilde{D}_1^{(0)}]\,, \\
\mathcal{F}_{LL} &= \mathcal{P}[\tilde{g}_{1L}^{(0)}\ \tilde{D}_1^{(0)}]\,, \\
\mathcal{F}_{LT}^{\cos(\phi_h-\phi_s)} &= \mathcal{P}[\tilde{g}_{1T}^{(1)}\ \tilde{D}_1^{(0)}]\,, \\
\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} &= \mathcal{P}[\tilde{h}_1^{(0)}\ \tilde{H}_1^{\perp(1)}]\,, \\
\mathcal{F}_{UU}^{\cos(2\phi_h)} &= \mathcal{P}[\tilde{h}_1^{\perp(1)}\ \tilde{H}_1^{\perp(1)}]\,, \\
\mathcal{F}_{UL}^{\sin(2\phi_h)} &= \mathcal{P}[\tilde{h}_{1L}^{\perp(1)}\ \tilde{H}_1^{\perp(1)}]\,, \\
\mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)} &= \frac{1}{4}\mathcal{P}[\tilde{h}_{1T}^{\perp(2)}\ \tilde{H}_1^{\perp(1)}].
\end{aligned}
$$

$$
\mathcal{P}[\tilde{f}^{(n)}\tilde{D}^{(m)}] \equiv x_B \sum e_a^2\,(zM|\boldsymbol{b}_T|)^n\,(zM_h|\boldsymbol{b}_T|)^m\,\tilde{f}^{a(n)}(x, z^2\boldsymbol{b}_T^2)\,\tilde{D}^{a(m)}(z, \boldsymbol{b}_T^2)
$$

# “BW in Generalized Parton Model”

Bessel weighting-$\mathcal{P}$rojecting Sivers orthogonality Bessel Fncts.

$$\mathcal{W} = \sin(\phi_h - \phi_S)\frac{2\,J_1(|\boldsymbol{P}_{hT}|\mathcal{B}_T)}{zM\mathcal{B}_T}$$

$$A_{UT}^{\frac{\mathcal{J}_1^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)}{zM}\sin(\phi_h-\phi_S)}(\mathcal{B}_T) \quad =$$

$$2\,\frac{\int d|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{P}_{h\perp}|\,d\phi_h\,d\phi_S\;\frac{\mathcal{J}_1^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)}{zM}\sin(\phi_h-\phi_S)\left(d\sigma^\uparrow - d\sigma^\downarrow\right)}{\int d|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{P}_{h\perp}|\,d\phi_h\,d\phi_S\;\mathcal{J}_0^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)\;\left(d\sigma^\uparrow + d\sigma^\downarrow\right)}$$

$$A_{UT}^{\frac{\mathcal{J}_1^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)}{zM}\sin(\phi_h-\phi_s)}(\mathcal{B}_T) = -2\,\frac{\sum_a e_a^2\,\tilde{f}_{1T}^{\perp(1)a}(x,z^2\mathcal{B}_T^2)\,\tilde{D}_1^a(z,\mathcal{B}_T^2)}{\sum_a e_a^2\;\tilde{f}_1^a(x,z^2\mathcal{B}_T^2)\,\tilde{D}_1^a(z,\mathcal{B}_T^2)}$$

# Traditional weighted asymmetry recovered ... but naively UV divergent

$$\lim_{\mathcal{B}_T \to 0} w_1 = 2J_1(|\boldsymbol{P}_{h\perp}|\mathcal{B}_T)/zM\mathcal{B}_T \longrightarrow |\boldsymbol{P}_{h\perp}|/zM$$

$$A_{UT}^{\frac{|\boldsymbol{P}_{h\perp}|}{z_h M}\sin(\phi_h - \phi_s)} = -2\frac{\sum_a e_a^2\ f_{1T}^{\perp(1)}(x)\ D_1^{a(0)}(z)}{\sum_a e_a^2\ f_1^{a(0)}(x)\ D_1^{a(0)}(z)}$$

Bacchetta et al. JHEP 08

***undefined w/o regularization***

# Part 2

- Impact on studying BW and TMD evolution

- Explore impact these BWA have on studying the <u>scale dependence</u> of the SIDIS cross section at <u>small to moderate transverse momentum</u> where the TMD framework is expected to give a good description of the cross section **Boer, Gamberg, Musch,Prokudin JHEP**

- **SKETCH TMD EVOLUTION ....**

- ★ The usefulness of Fourier-Bessel transforms in studying the factorization as well as the scale dependence of transverse momentum dependent cross section has been known for over 30 years.
- ★ Is the natural language for TMD Evolution

★ Collins Soper (81), Collins, Soper, Sterman (85), Boer (01) (09) (13), Ji,Ma,Yuan (04), Collins-Cambridge University Press (11), Aybat Rogers PRD (11), Abyat, Collins, Qiu, Rogers (11), Aybat, Prokudin, Rogers (11), Bacchetta, Prokudin (13), Sun, Yuan (13), Aidala, Field, Gamberg, Rogers (14),Collins Rogers 2015 ….

# Studies that impact TMD Factorization

## Fixed scale phenomenology- Stage 1+

A.V. Efremov, K. Goeke, S. Menzel, A. Metz, and P.
Schweitzer, Phys. Lett. B 612, 233 (2005).
W. Vogelsang and F. Yuan, Phys. Rev. D 72, 054028 (2005).
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# Comments

- Collins-Soper evol. kernel has perturbative-short distance & non-perturbative (**NP**) large-distance content
- **Non-pertb. large-distance is *strongly universal* -many interesting predictions**
- Universal character can exploited in observables "Bessel Weighting"

(Boer Gamberg, Musch Prokudin JHEP 2011, Aghasyan, Avakian, Gamberg, Prokudin, Rossi et al 2014)

# TMD factorized cross section SIDIS

$$
\begin{aligned}
\frac{d\sigma}{dP_T^2} \propto \sum_{jj'} \mathcal{H}_{jj',\,\mathrm{SIDIS}}(\alpha_s(\mu),\mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T\cdot\boldsymbol{P}_T}\,\tilde{F}_{H_1}(x,b_*;\mu_b,\mu_b^2)\tilde{D}_{H_2}(z,b_*;\mu_b,\mu_b^2) \\
\exp\Bigg\{ - g_{\mathrm{PDF}}(x,b_T;b_{\max}) - g_{\mathrm{FF}}(z,b_T;b_{\max}) - 2g_K(b_T;b_{\max})\ln\left(\frac{Q}{Q_0}\right) \\
+ 2\ln\left(\frac{Q}{\mu_b}\right)\tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{Q}\frac{d\mu'}{\mu'}\left[\gamma_{\mathrm{PDF}}(\alpha_s(\mu');1) + \gamma_{\mathrm{FF}}(\alpha_s(\mu');1) - 2\ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\Bigg\} \\
+ Y_{\mathrm{SIDIS}}\,. \quad + P.S.C
\end{aligned}
$$

Collins 2011 (Cambridge Univ. Press)

# Elements of TMD Fact. Cross section

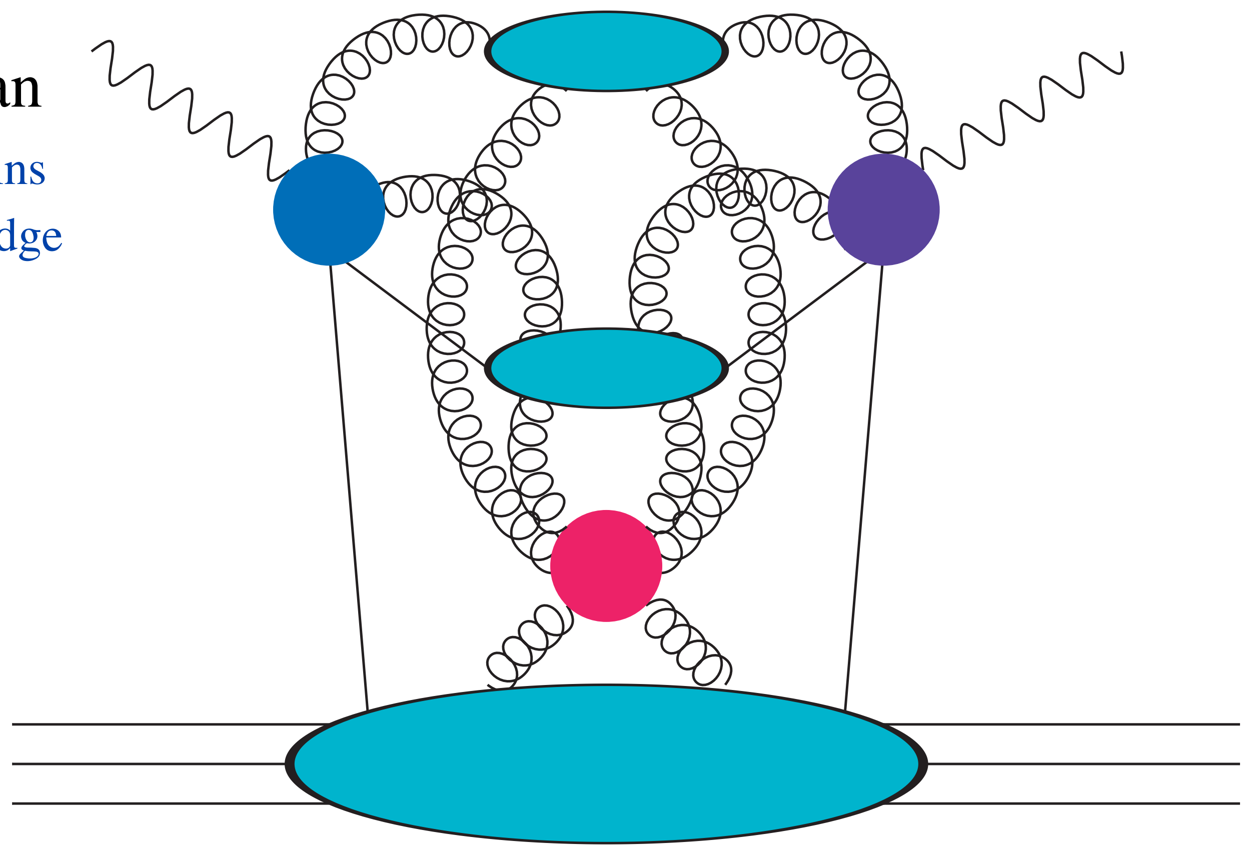
- $Y$ term serves to correct expression for structure function when $P_T \sim Q$

- Exponent contains both perturbative and non-perturbative content arising from TMD factorization $\longleftrightarrow$ evolution

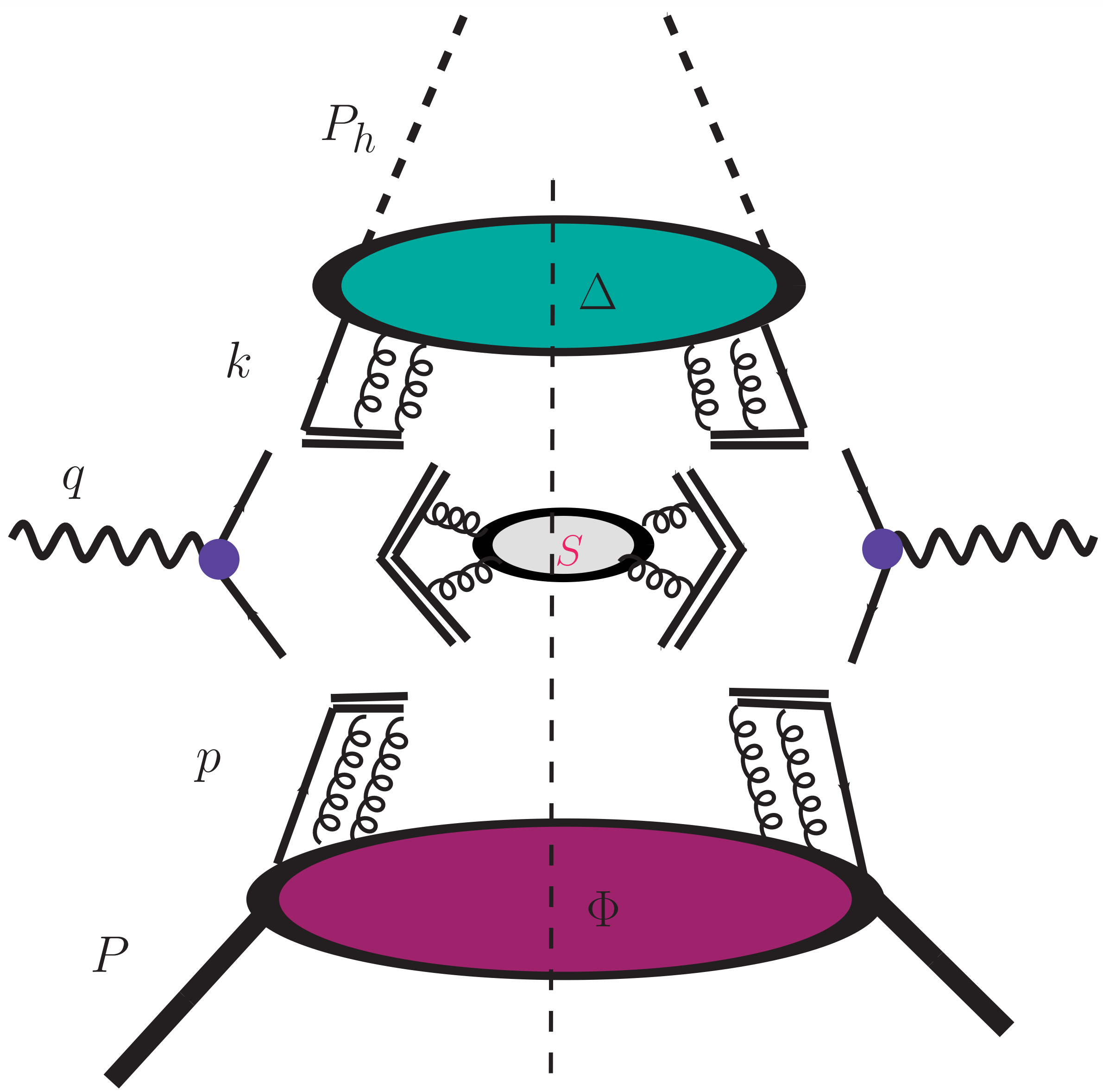
- Where does this structure come from ... of course this is based upon earlier CS 81 & CSS 85 formalism but new treatment of soft factor and CSS equations effectively implements "resummation" of large logs.

# TMD Factorization Procedure while maintaining a Parton Model picture: CSS + JC 2011 To study nucleon structure

- Leading Regions-power counting Libby Sterman PRD 1978 (see Collins PRD 1980 nongauge theories, Collins Soperp NPB& CSS formalism 1982-85… Collins 2011 Cambridge Univ. Press)
- "Reduced Diagrams"
- Apply Ward Identities get factorized form
  - Soft Factor w/ gauge links & rapidity div
  - TMDs w/ gauge links & rapidity div
  - Hard contribution



# Further Beyond Parton Model "tree level" factorization




- TMDs w/Gauge links: color invariant
- *In addition Soft factor*

- Extra divergences at one loop and higher
- Extra parameters needed to regulate light-cone divergences, soft & collinear divergences
- Modifies convolution integral introduction of soft factor
- Some effects of evolution cancel in Bessel weighted asymmetries

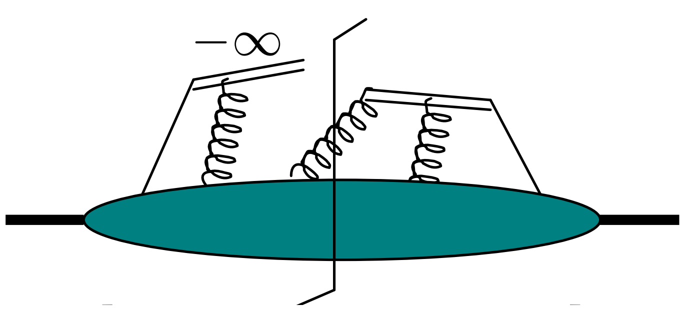
# Further treatment achieve full factorization using Soft Factor in CSS

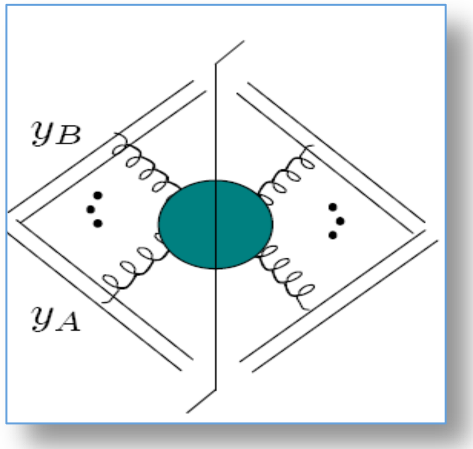
- Lightlike Wilson lines in TMDs $W(\infty, x; n) = P\exp\left[-ig_0\int_0^\infty ds n\cdot A_0^a(x+sn)t^a\right].$
  - Infinite rapidity QCD radiation in the wrong direction.
  - In soft factor/fragmentation function too.

$$y_B = \frac{1}{2}\ln\left(\frac{n^+}{n^-}\right)$$

$$n^- = 0 \rightarrow$$

$$\lim y_B \rightarrow -\infty$$



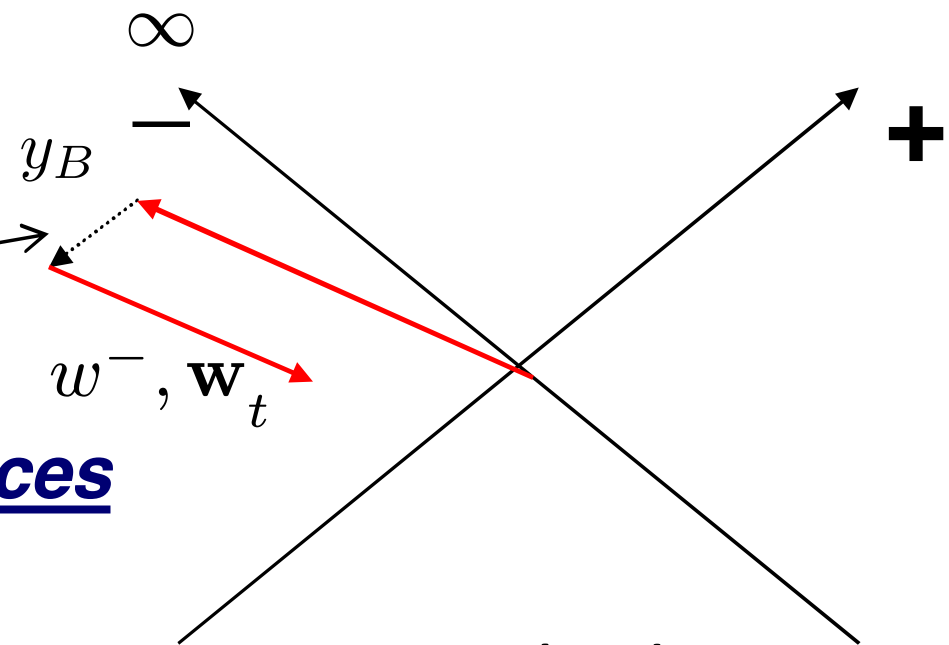

- Finite rapidity Wilson lines
  - Regulate rapidity of extra gluons.

$$n^- = (-e^{2y_B}, 1, \mathbf{0})$$

# Introduce rapidity scale parameter to regulate LC Divergences arising in Gauge link and soft factor

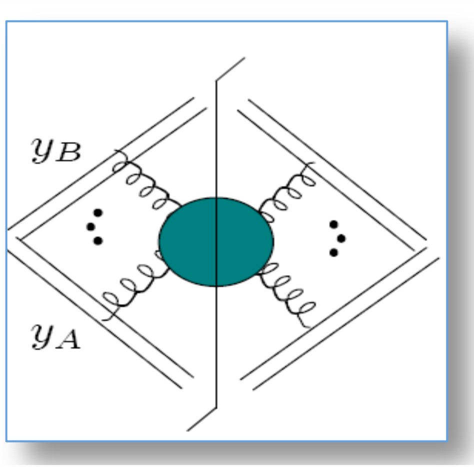
$$n_{\rm B} = (-e^{2y_{\rm B}}, 1, \mathbf{0}_t).$$

***Tilt to regulate rapidity divergences***




*TMD tilted*

Collins Act Pol. 2003
Ji Ma Yuan 2004, 2005




$\Longleftrightarrow \quad y$

# Emergence of Soft Factor in TMDs

Start with only the hard part factorized:

$$d\sigma = |\mathcal{H}|^2 \, \frac{\tilde{F}_1^{\rm unsub}(y_1 - (-\infty)) \times \tilde{F}_2^{\rm unsub}(+\infty - y_2)}{\tilde{S}(+\infty, -\infty)}$$

Soft factor repartitioned
This is done to both

1) cancel LC divergences and
2) separate "right & left" movers i.e. factorize

$$d\sigma = |\mathcal{H}|^2 \, \left\{ F_1^{\rm unsub}(y_1 - (-\infty)) \sqrt{\frac{\tilde{S}(+\infty, y_s)}{\tilde{S}(+\infty, -\infty)\tilde{S}(y_s, -\infty)}} \right\} \times \left\{ \tilde{F}_2^{\rm unsub}(+\infty - y_2) \sqrt{\frac{\tilde{S}(y_s, -\infty)}{\tilde{S}(+\infty, -\infty)\tilde{S}(+\infty, y_s)}} \right\}$$

*Separately Well-defined*

$$\frac{d\sigma}{dP_T^2} \propto \sum_{jj'} \mathcal{H}_{jj',\,\mathrm{SIDIS}}(\alpha_s(\mu), \mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T\cdot\boldsymbol{P}_T}\ \tilde{F}_{j/H_1}(x, b_T; \mu, \zeta_1)\, \tilde{D}_{H_2/j'}(z, b_T; \mu, \zeta_2) \ + \ Y_{\mathrm{SIDIS}}$$

In full QCD, the auxiliary parameters are exactly arbitrary and this is reflected in the the Collins-Soper (CS) equations for the TMD PDF, and the renormalization group (RG) equations

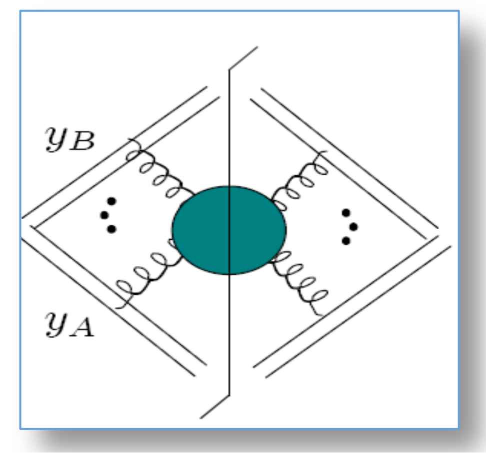
Collins arXiv: 1212.5974

# *Evolution follows from their independence of rapidity scale*

$$\tilde{F}_H^{\mathrm{sub}}(x, b_T; \mu, y_n) = \lim_{\substack{y_A \to \infty \\ y_B \to -\infty}} \tilde{F}_H^{\mathrm{unsub}}(x, b_T; \mu, y_P - y_B) \sqrt{\frac{\tilde{S}(b_T; y_A, y_n)}{\tilde{S}(b_T; y_A, y_B)\tilde{S}(b_T; y_n, y_B)}}$$

From operator definition get

Collins-Soper Equation:




$$-\quad \frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu)$$

$$\tilde{K}(b_T; \mu) = \frac{1}{2} \frac{\partial}{\partial y_n} \ln \frac{\tilde{S}(b_T; y_n, -\infty)}{\tilde{S}(b_T; +\infty, y_n)}$$

Soft factor further "repartitioned"
This is done to

1) cancel LC divergences in "unsubtracted" TMDs
2) separate "right & left" movers i.e. full factorization
3) remove double counting of momentum regions

# *Along with .... Renormalization group Equations*

$$\frac{d\tilde{K}}{d\ln\mu} = -\gamma_K(g(\mu))$$

$$\frac{d\ln\tilde{F}(x, b_T; \mu, \zeta)}{d\ln\mu} = -\gamma_F(g(\mu); \zeta/\mu^2)$$

RGE:
get anomalous
for *F* & *K*

Solve Collins Soper & RGE eqs. to obtain "evolved TMDs"

# Evolved TMDs

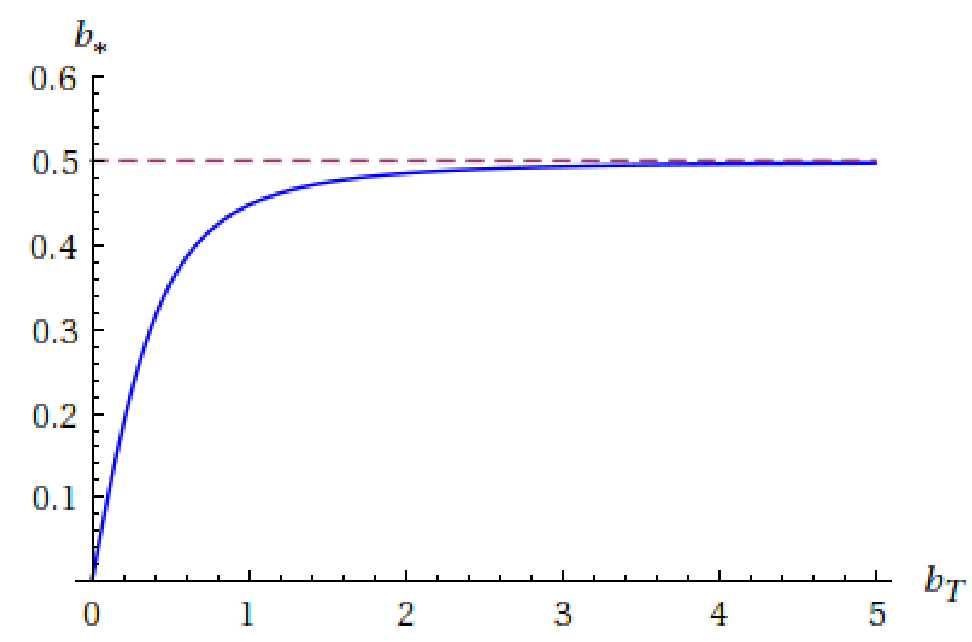
- John's talk one "segregates"
  small $b_T$ - Pert & Large $b_T$ -non-perturbative

# One TMD factorization entire range of $P_T$ or $b_T$

Collins Soper Sterman NPB 85

- Maximizes the perturbative content while providing a TMD formalism that is applicable over the entire range of $P_T$

$$\mathbf{b}_* = \frac{\mathbf{b}_T}{\sqrt{1 + b_T^2/b_{\max}^2}}, \qquad \mu_b = \frac{C_1}{b_*}.$$



# Non-perturbative part of $\tilde{K}(b_T, \mu)$

Solve RGE:

Collins Soper Sterman NPB 85

$$\tilde{K}(b_T; \mu) = \tilde{K}(b_*; \mu_b) - \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(g(\mu')) - g_K(b_T)$$

$$\mathbf{b}_* = \frac{\mathbf{b}_T}{\sqrt{1 + b_T^2/b_{\max}^2}}, \qquad \mu_b = \frac{C_1}{b_*}.$$

$b_{\max}$ chosen so that $b_*$ doesn't go too far beyond the pertb. region maximize perturbative content in evolving TMDs and cross section

# Structure Function & *TMDs in QCD*

$$\mathcal{F}_{UU}(x,z,b,Q^2) = \sum_a \tilde{F}^a_{H1}(x,b_T,\mu,\zeta_F)\tilde{D}^a_{H2}(z_h,b_T,\mu,\zeta_D)H_{UU}(Q^2,\mu^2)$$

# Evolved Structure Function & *TMDs in b-space*

$$\mathcal{F}_{UU}(x,z,b,Q^2) = \sum_a \tilde{F}^a_{H1}(x,b_T,\mu,\zeta_F)\tilde{D}^a_{H2}(z_h,b_T,\mu,\zeta_D)H_{UU}(Q^2,\mu^2)$$

**Non-perturbative large $b_T$ behavior**

**Totally universal related to derivative of soft factor independent of x & hadron**

$$\tilde{F}_{H_1}(x,b_T;Q,Q^2) = \tilde{F}_{H_1}(x,b_*;\mu_b,\mu_b^2)\exp\left\{-g_1(x,b_T;b_{\max}) - g_K(b_T;b_{\max})\ln\left(\frac{Q}{Q_0}\right) + \ln\left(\frac{Q}{\mu_b}\right)\tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{Q}\frac{d\mu'}{\mu'}\left[\gamma_{\rm PDF}(\alpha_s(\mu');1) - \ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\right\}$$

**perturbative small $b_T$ behavior**

These functions have good perturbative behavior at entire range of $b_T$

## Unpolarized and Sivers evolve in same way !!!

Recall correlator in *b*-space From Bessel Transform

$$\tilde{\Phi}^{[\gamma^+]}(x,\boldsymbol{b}_T) \;=\; \tilde{f}_1(x,\boldsymbol{b}_T^2) - i\,\epsilon_T^{\rho\sigma} b_{T\rho} S_{T\sigma}\, M\,\tilde{f}_{1T}^{\perp(1)}(x,\boldsymbol{b}_T^2)$$

$$\frac{\partial\,\tilde{\phi}^i_{f/P}(x,\mathbf{b}_\mathrm{T};\mu,\zeta_F)\epsilon_{ij}S_T^j}{\partial\ln\sqrt{\zeta_F}} = \tilde{K}(b_T;\mu)\tilde{\phi}^i_{f/P}(x,\mathbf{b}_\mathrm{T};\mu,\zeta_F)\epsilon_{ij}S_T^j.$$

Collins Soper Equation

# Sivers Structure Function

$$\mathcal{F}_{UT}(x,z,b,Q) = \tilde{f}^{(1)}_{1T\,i/P}(x,b_\star;\mu_b)\tilde{D}_{H/j}(z,b_\star;\mu_b)e^{-S^{pert}(b_*,Q)}e^{-S^{NP}_{UT}(b,Q,x,z)}H_{UT}$$

★ Abyat, Collins, Qiu, Rogers PRD (11), $b_* = \frac{b}{\sqrt{1+(b/b_{max})^2}}$

$$e^{-S^{NP}_{UT}}(b,Q,x,z) = \exp\left\{-\left[g_1(x,b_T;b_{\max}) + g_2(z,b_T;b_{\max}) + 2g_k(b_T)\ln\left(\frac{Q}{Q_0}\right)\right]\right\}_{UT}$$

Non perturbative factor contribution must be fit

CSS NPB 85

# Sivers BWA: Cancellation of Universal *NP* and flavor blind hard contributions

When $\Lambda^2_{QCD} \ll P_h^2 \ll Q^2$

$$\mathcal{A}_{UT}(x,z,b,Q^2)$$
$$= \frac{\tilde{f}_{1T}^{\perp(1)}(x,z^2\boldsymbol{b}^2,\mu_0^2,Q_0)\tilde{D}_1(z_h,\boldsymbol{b}^2,\mu_0^2,Q_0)\tilde{H}_{UT}(\mu_0^2,Q_0)e^{-S^{\mathrm{pert}}(b_*,Q)}e^{-2g_k(b_T)\ln\left(\frac{Q}{Q_0}\right)}}{\tilde{f}_1(x,z^2\boldsymbol{b}^2,\mu_0^2,Q_0)\tilde{D}_1(z_h,\boldsymbol{b}^2,\mu_0^2,Q_0)\tilde{H}_{UU}(\mu_0^2,Q_0)e^{-S^{\mathrm{pert}}(b_*,Q)}e^{-2g_k(b_T)\ln\left(\frac{Q}{Q_0}\right)}}$$

**BWA less sensitivity to TMD Evolution**
**Prediction of TMD factorization & Evolution**

Boer, Gamberg, B. Musch, A. Prokudin....

# Bessel Weighting of experimental observables

- What good is all of this?
- Test the idea
- How?
- We used a MC
- So first re-write BWA for an “experiment”

# Part 3

## A First pheno study of BW of Experimental Observables

**Studies of transverse momentum dependent parton distributions and Bessel weighting** JHEP *2015*

**M. Aghasyan,**[a,b] **H. Avakian,**[c] **E. De Sanctis,**[a] **L. Gamberg,**[d] **M. Mirazita,**[a] **B. Musch,**[e] **A. Prokudin**[c] **and P. Rossi**[a,c]

New experimental tool to study the 3-D nucleon content to the SIDIS cross section that minimizes the transverse momentum model dependencies inherent in conventional extractions of TMDs.

# So lets consider the Bessel Weighted double spin Asymm in *b*-space

$$S_{||}\lambda_e = \pm 1$$

# Where the Parton Model Structure Functions in *b*-space are ...

$$\mathcal{F}_{UU,T} = x \sum_a e_a^2 \tilde{f}_1^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2)\,,\; \mathcal{F}_{LL} = x \sum_a e_a^2 \tilde{g}_{1L}^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2)$$

# Where the Parton Model Structure Functions in *b*-space are ...

$$\mathcal{F}_{UU,T} = x\sum_a e_a^2 \tilde{f}_1^a(x, z^2 b_T^2)\tilde{D}_1^a(z, b_T^2)\,, \ \mathcal{F}_{LL} = x\sum_a e_a^2 \tilde{g}_{1L}^a(x, z^2 b_T^2)\tilde{D}_1^a(z, b_T^2)$$

## Remind ourselves of Asymmetry in "*b*" space for double longitudinal polarized process

$$S_{||}\lambda_e = \pm 1$$

So Bessel Weighted double spin Asymm in *b*-space

$$A_{LL}^{J_0(b_T P_{h\perp})}(b_T) = \frac{\tilde{\sigma}^+(b_T) - \tilde{\sigma}^-(b_T)}{\tilde{\sigma}^+(b_T) + \tilde{\sigma}^-(b_T)} \equiv \frac{\tilde{\sigma}_{LL}(b_T)}{\tilde{\sigma}_{UU}(b_T)} = \sqrt{1-\varepsilon^2}\,\frac{\sum_a e_a^2 \tilde{g}_{1L}^a(x, z^2 b_T^2)\tilde{D}_1^a(z, b_T^2)}{\sum_a e_a^2 \tilde{f}_1^a(x, z^2 b_T^2)\tilde{D}_1^a(z, b_T^2)}$$

★ First we project out the Structure functions going into asymmetry from Multipole expansion

$$
\begin{aligned}
&\frac{d\sigma}{dx_B\, dy\, d\phi_S\, dz_h\, d\phi_h\, |\boldsymbol{P}_{h\perp}| d|\boldsymbol{P}_{h\perp}|} = \\
&\frac{\alpha^2}{x_B y Q^2} \frac{y^2}{(1-\varepsilon)} \left(1+\frac{\gamma^2}{2x_B}\right) \int \frac{d|\boldsymbol{b}_T|}{(2\pi)} |\boldsymbol{b}_T| \Big\{ J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UU,T} + \varepsilon\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UU,L} \\
&+ \quad \sqrt{2\,\varepsilon(1+\varepsilon)}\, \cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UU}^{\cos\phi_h} + \varepsilon \cos(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UU}^{\cos(2\phi_h)} \\
&+ \lambda_e \sqrt{2\,\varepsilon(1-\varepsilon)}\, \sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LU}^{\sin\phi_h} \\
&+ \quad S_\parallel \left[ \sqrt{2\,\varepsilon(1+\varepsilon)}\, \sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UL}^{\sin\phi_h} + \varepsilon \sin(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UL}^{\sin 2\phi_h} \right] \\
&+ S_\parallel \lambda_e \left[ \sqrt{1-\varepsilon^2}\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LL} + \sqrt{2\,\varepsilon(1-\varepsilon)}\, \cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LL}^{\cos\phi_h} \right] \\
&+ |\boldsymbol{S}_\perp| \Big[ \sin(\phi_h-\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|) \left( \mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon\, \mathcal{F}_{UT,L}^{\sin(\phi_h-\phi_S)} \right) \\
&\qquad + \varepsilon\, \sin(\phi_h+\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} \\
&\qquad + \varepsilon\, \sin(3\phi_h-\phi_S)\, J_3(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)} \\
&\qquad + \sqrt{2\,\varepsilon(1+\varepsilon)}\, \sin\phi_S\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UT}^{\sin\phi_S} \\
&\qquad + \sqrt{2\,\varepsilon(1+\varepsilon)}\, \sin(2\phi_h-\phi_S)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{UT}^{\sin(2\phi_h-\phi_S)} \Big] \\
&+ |\boldsymbol{S}_\perp| \lambda_e \Big[ \sqrt{1-\varepsilon^2}\, \cos(\phi_h-\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LT}^{\cos(\phi_h-\phi_S)} \\
&\qquad + \sqrt{2\,\varepsilon(1-\varepsilon)}\, \cos\phi_S\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LT}^{\cos\phi_S} \\
&\qquad + \sqrt{2\,\varepsilon(1-\varepsilon)}\, \cos(2\phi_h-\phi_S)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\, \mathcal{F}_{LT}^{\cos(2\phi_h-\phi_S)} \Big] \Big\} \qquad .)
\end{aligned}
$$

# Project the Structure functions from differential cross section

$$d\Phi \equiv dx\,dy\,d\psi\,dz\,dP_{h\perp}P_{h\perp}$$

$$\int dP_{h\perp}P_{h\perp}J_0(b_T P_{h\perp})\left(\frac{d\sigma^+}{d\Phi}+\frac{d\sigma^-}{d\Phi}\right) = K(x,y)\mathcal{F}_{UU,T}$$

$$\int dP_{h\perp}P_{h\perp}J_0(b_T P_{h\perp})\left(\frac{d\sigma^+}{d\Phi}-\frac{d\sigma^-}{d\Phi}\right) = K(x,y)\sqrt{1-\varepsilon^2}\mathcal{F}_{LL}$$

# Let us re-write cross section in terms of events

$$\int dP_{h\perp} P_{h\perp} J_0(b_T P_{h\perp}) \left( \frac{1}{\mathcal{N}_0^+} \frac{dn^+}{d\Phi} + \frac{1}{\mathcal{N}_0^-} \frac{dn^-}{d\Phi} \right) = K(x,y) \mathcal{F}_{UU,T}$$

$$\int dP_{h\perp} P_{h\perp} J_0(b_T P_{h\perp}) \left( \frac{1}{\mathcal{N}_0^+} \frac{dn^+}{d\Phi} - \frac{1}{\mathcal{N}_0^-} \frac{dn^-}{d\Phi} \right) = K(x,y) \sqrt{1-\varepsilon^2} \mathcal{F}_{LL}$$

$dn^{\pm}$ are the number of events in a differential phase space volume, $d\Phi$, and $\mathcal{N}_o^{\pm}$ is the standard normalization factor, that is the product of the number of beam and target particles with ± polarization per unit target area. We assume that the experiment has been set up such that $\mathcal{N}_o^+ = \mathcal{N}_o^-$.

# Next discretize differential cross section

$$d\Phi \equiv dx\, dy\, d\psi\, dz\, dP_{h\perp} P_{h\perp} \longrightarrow \Delta\Phi \equiv \Delta x\, \Delta y\, \Delta z\, \Delta P_{h\perp} P_{h\perp}$$

And re-do/reconsider the projecting e.g.

$$\int dP_{h\perp} P_{h\perp} J_0(B_T P_{h\perp}) \frac{dn^{\pm}}{d\Phi} \Longrightarrow \sum_{i\,\epsilon\,\mathrm{bin}[x,y,z]} J_0(B_T P_{h\perp\, i}) \frac{\Delta n^{\pm}}{\Delta x\, \Delta y\, \Delta z}$$

# Sum over events in bin to sum over events

$$K(x,y)\,\sqrt{1-\varepsilon^2}\,\mathcal{F}_{LL}(B_T) =$$

$$\Rightarrow \left\{ \sum_{j\ \text{events}}^{N^+} J_0(B_T P_{h\perp\,j}) \quad - \quad \sum_{j\ \text{events}}^{N^-} J_0(B_T P_{h\perp\,j}) \right\}$$

# Experimental procedure to BWA for double longitudinal beam/target polarization

$$A_{LL}^{J_0(b_T P_{h\perp})}(b_T) = \frac{\tilde{\sigma}^+(b_T) - \tilde{\sigma}^-(b_T)}{\tilde{\sigma}^+(b_T) + \tilde{\sigma}^-(b_T)}$$

$$= \frac{\sum_j^{N^+} J_0(b_T P_{h\perp j}^{[+]}) - \sum_j^{N^-} J_0(b_T P_{h\perp j}^{[-]})}{\sum_j^{N^+} J_0(b_T P_{h\perp j}^{[+]}) + \sum_j^{N^-} J_0(b_T P_{h\perp j}^{[-]})} \equiv \frac{\tilde{S}^+ - \tilde{S}^-}{\tilde{S}^+ + \tilde{S}^-}$$

$j$ are indices for the sums on events and $N^{\pm}$ are the number of events, for positive/negative products of lepton and nucleon helicities and at given $x$, $y$ and $z$, and where $S^{\pm}$ indicate the sum over events for ± helicities.

# Method....

- Every time you have an event at a $P_h$ plug in the value of P and get a value for, $J_n(b\ P_h)$ and then perform the sums
- Test this idea w/ a Monte Carlo

# Developed a differential Monte Carlo based on parton model to test the Bessel Weighting

$$\ell(l) + N(P) \to \ell(l') + h(P_h) + X,$$

Kotzinian NPB 1995
Mulders Tangerman NPB 1996
Bacchetta et al. JHEP 2006
Anselmino et al. PRD 71 2005

$$\frac{d\sigma}{dxdydzd^2\boldsymbol{p}_\perp d^2\boldsymbol{k}_\perp d\phi_{l'}} = 2\,K(x,y)J(x,Q^2,\boldsymbol{k}_\perp^2) \times x\sum_a e_a^2\left[f_{1,a}(x,\boldsymbol{k}_\perp^2)D_{1,a}(z,\boldsymbol{p}_\perp^2) + \lambda\sqrt{1-\varepsilon^2}g_{1L,a}(x,\boldsymbol{k}_\perp^2)D_{1,a}(z,\boldsymbol{p}_\perp^2)\right]$$

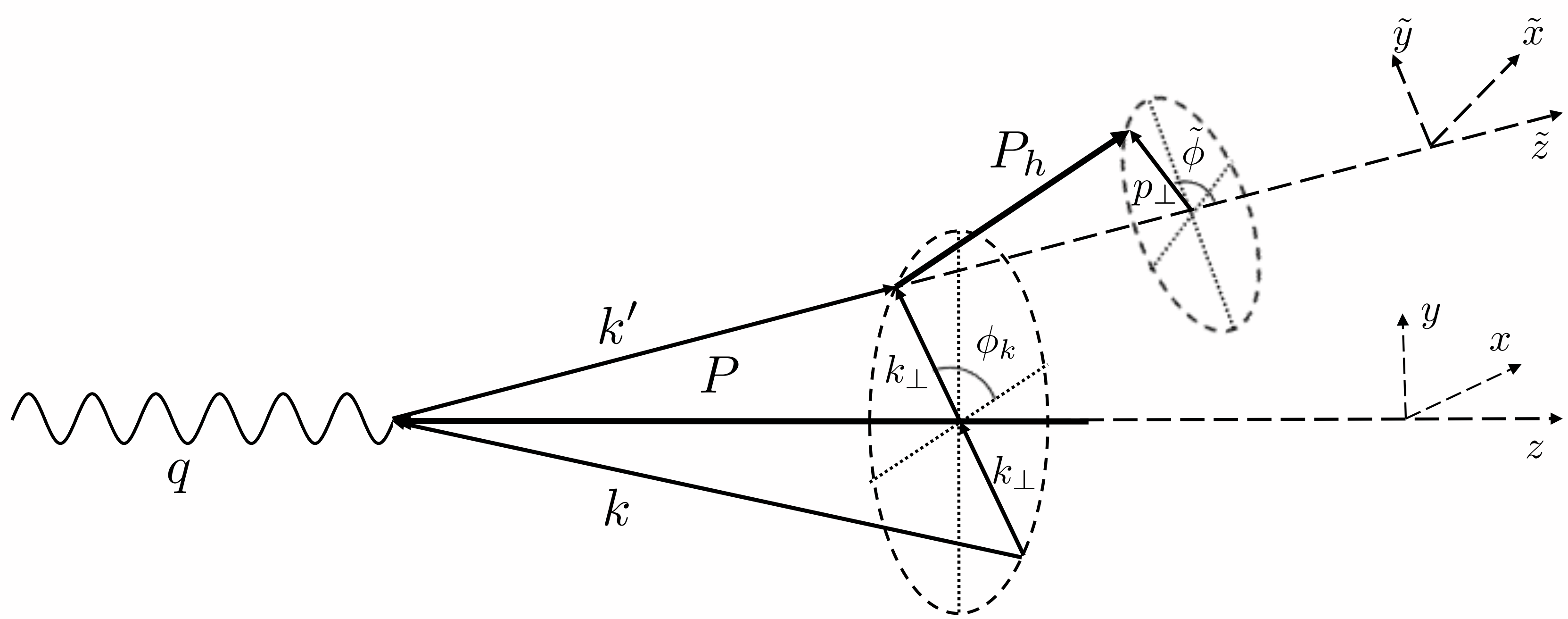



**Figure 1**. Kinematics of the process. $q$ is the virtual photon, $k$ and $k'$ are the initial and struck quarks, $k_\perp$ is the quark transverse component. $P_h$ is the final hadron with a $p_\perp$ component, transverse with respect to the fragmenting quark $k'$ direction.

# Input distributions to MC

$$f_1(x, \boldsymbol{k}_\perp^2) = f_1(x) \frac{1}{\langle k_\perp^2(x) \rangle_{f_1}} \exp\left( - \frac{\boldsymbol{k}_\perp^2}{\langle k_\perp^2(x) \rangle_{f_1}} \right) , \tag{3.9}$$

$$g_{1L}(x, \boldsymbol{k}_\perp^2) = g_{1L}(x) \frac{1}{\langle k_\perp^2(x) \rangle_{g_1}} \exp\left( - \frac{\boldsymbol{k}_\perp^2}{\langle k_\perp^2(x) \rangle_{g_1}} \right) , \tag{3.10}$$

$$D_1(z, \boldsymbol{p}_\perp^2) = D_1(z) \frac{1}{\langle p_\perp^2(z) \rangle} \exp\left( - \frac{\boldsymbol{p}_\perp^2}{\langle p_\perp^2(z) \rangle} \right) , \tag{3.11}$$

$$\langle k_\perp^2(x) \rangle = C\, x(1-x) \qquad \langle p_\perp^2(z) \rangle = D\, z(1-z)$$

$$C = 0.54\,\mathrm{GeV}^2 \ \text{ and } \ D = 0.5\,\mathrm{GeV}^2.$$

The generator we construct is implemented with on-shell initial partons with four momentum conservation imposed.

The limitations due to available phase space integration will modify the reconstructed distributions with respect to the input

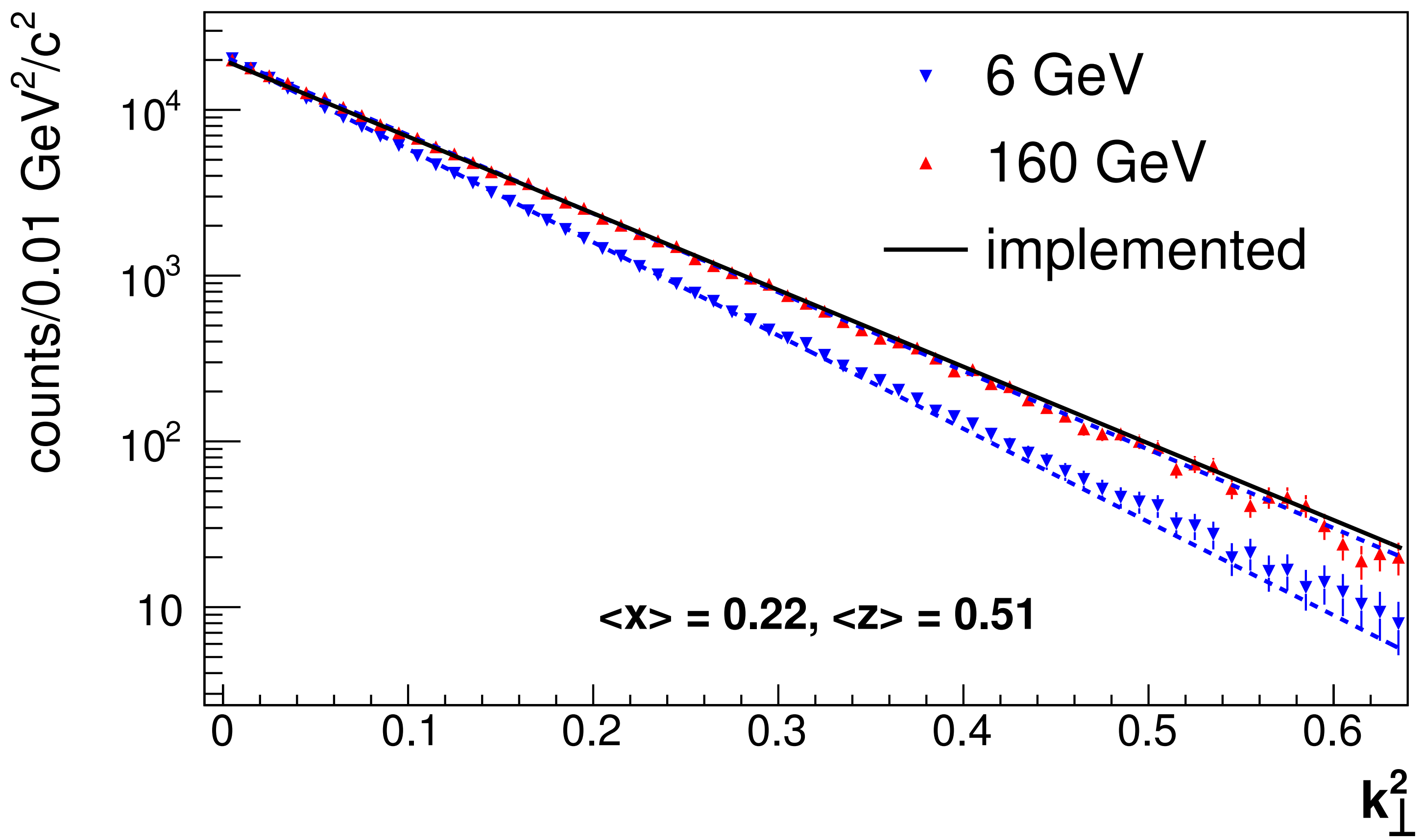
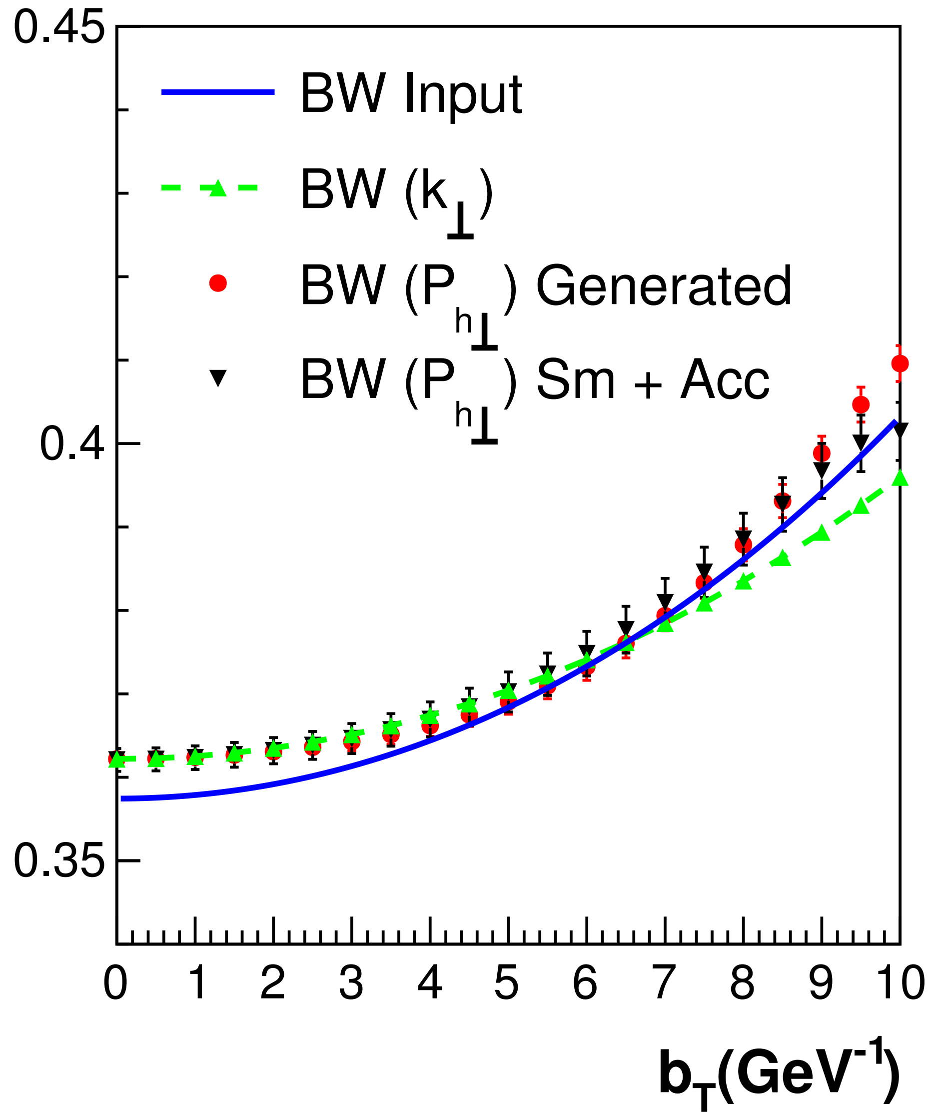
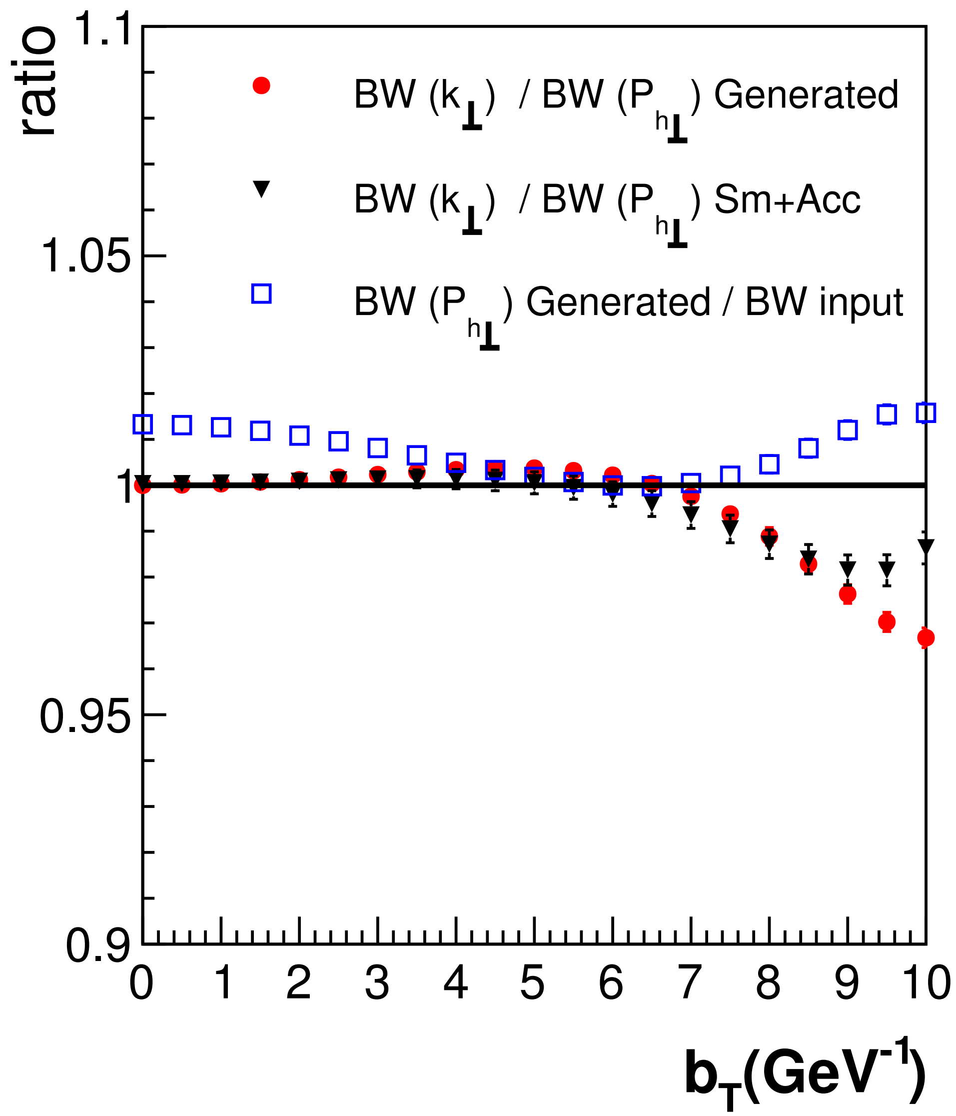



**Figure 2**. (Color online) The solid line is the Gaussian input distribution implemented using eq. (3.9), with red triangles coming from the Monte Carlo at 160 GeV initial lepton energy, blue triangles coming from the Monte Carlo at 6 GeV. The dashed line represents the fit to the Monte Carlo distributions which returned values of $C = 0.527\,\mathrm{GeV}^2$ and $C = 0.444\,\mathrm{GeV}^2$ at 160 GeV and 6 GeV respectively.

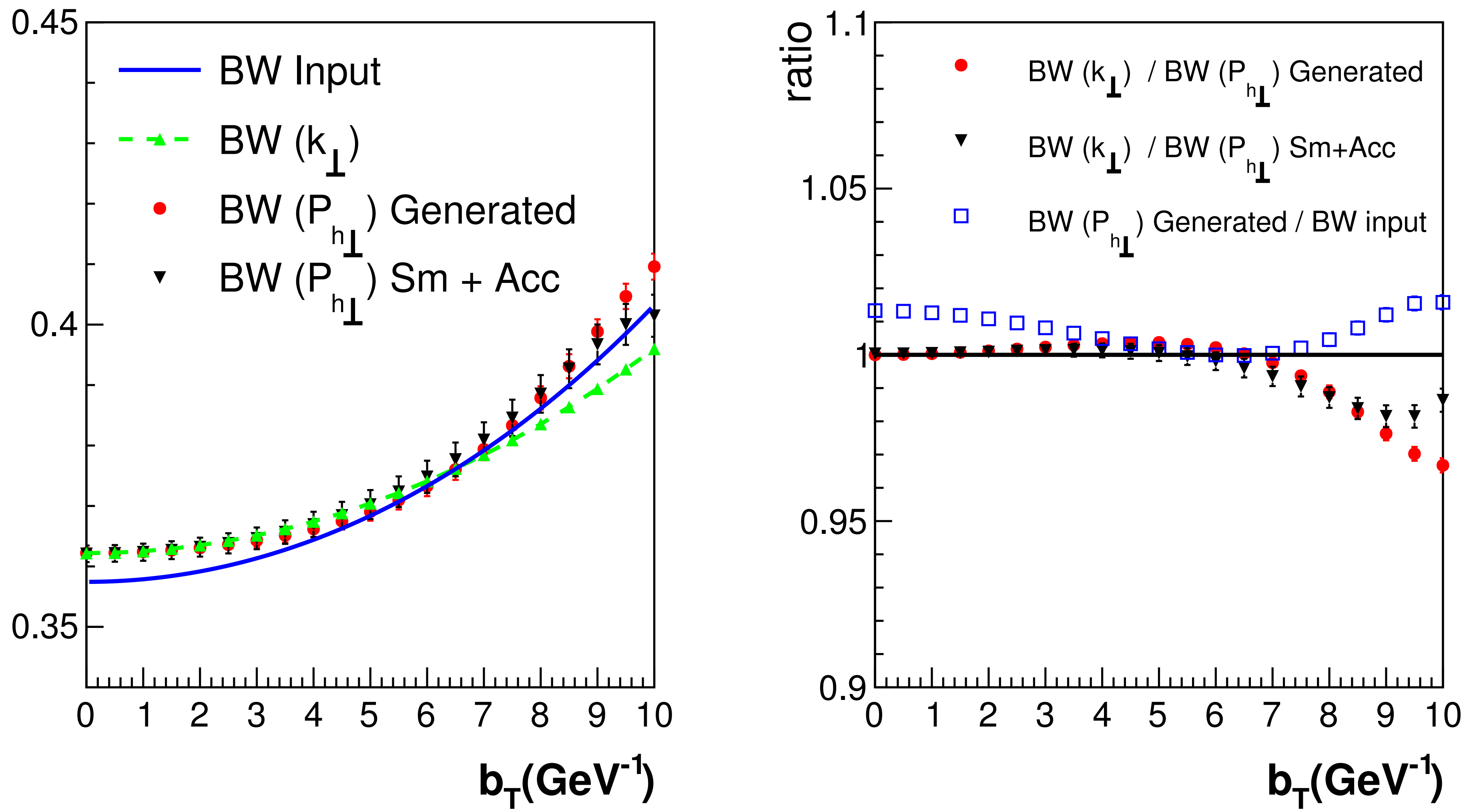
# well ...????



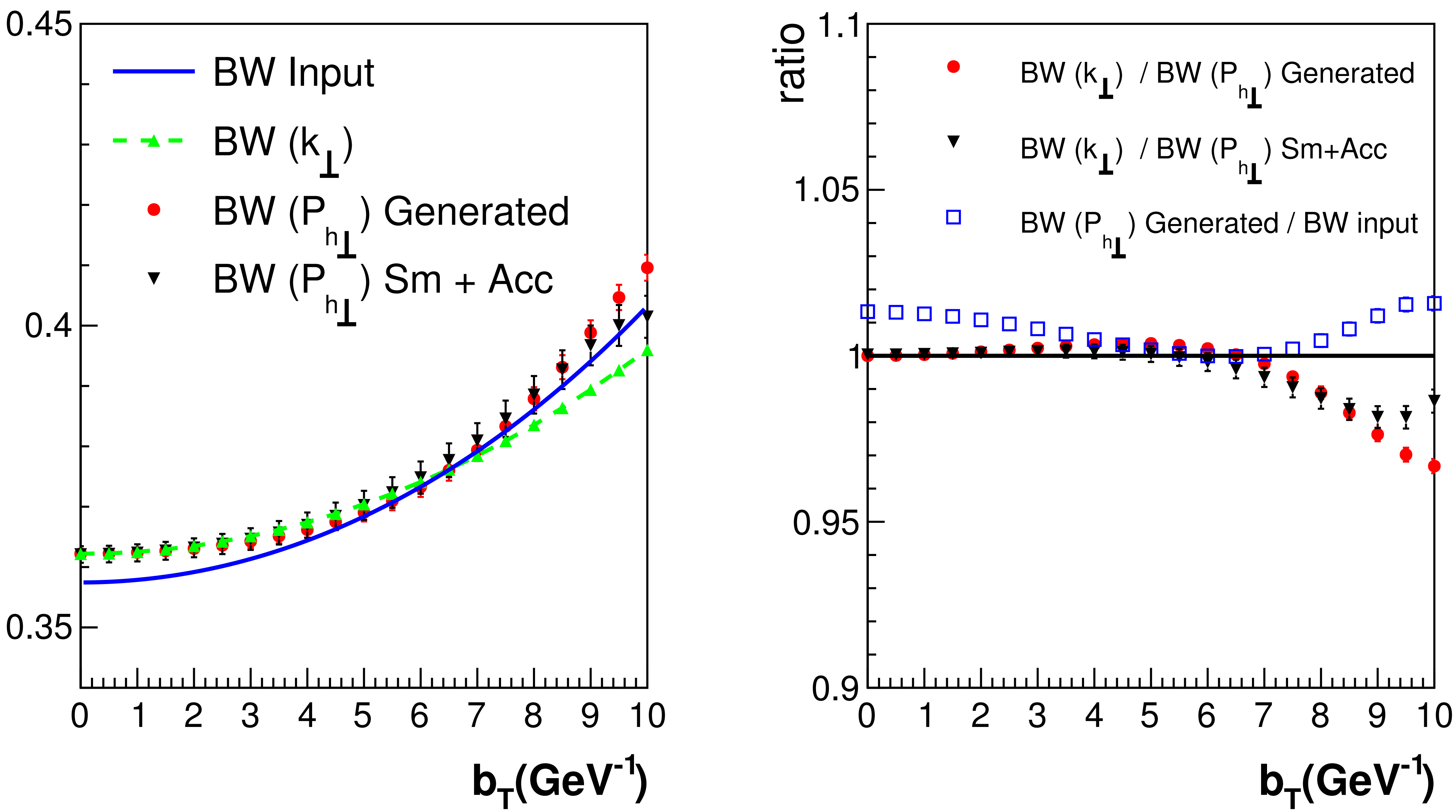

$$A_{LL}^{J_0(b_T P_{h\perp})}(b_T) = \sqrt{1-\varepsilon^2}\,\frac{\sum_a e_a^2 \tilde{g}_{1L}^a(x, z^2 b_T^2)\tilde{D}_1^a(z, b_T^2)}{\sum_a e_a^2 \tilde{f}_1^a(x, z^2 b_T^2)\tilde{D}_1^a(z, b_T^2)}$$

The blue curve labeled "BW Input", is the asymmetry calculated analytically using the right hand side of Eq and the Fourier transformed input distribution functions




Compare w/ the Monte Carlo generated distribution using Eq (full red points) labeled "BW(Ph⊥) Generated",

$$A_{LL}^{J_0(b_T P_{h\perp})}(b_T) = \frac{\sum_j^{N^+} J_0(b_T P_{h\perp j}^{[+]}) - \sum_j^{N^-} J_0(b_T P_{h\perp j}^{[-]})}{\sum_j^{N^+} J_0(b_T P_{h\perp j}^{[+]}) + \sum_j^{N^-} J_0(b_T P_{h\perp j}^{[-]})}$$




Compare w/ the Monte Carlo generated distribution using Eq (green) labeled "BW(k⊥) Generated",

$$a_{LL}^{J_0(b_T k_\perp)}(b_T) \equiv \quad \sqrt{1-\varepsilon^2}\frac{\tilde{g}_{1L}(b_T)}{\tilde{f}_1(b_T)} = \frac{\sum_j^{N^+} J_0(b_T k_{\perp j}^{[+]}) - \sum_j^{N^-} J_0(b_T k_{\perp j}^{[-]})}{\sum_j^{N^+} J_0(b_T k_{\perp j}^{[+]}) + \sum_j^{N^-} J_0(b_T k_{\perp j}^{[-]})}$$

# Bo-Qiang Ma and Zhun Lu PRD 87 2013 model calculation

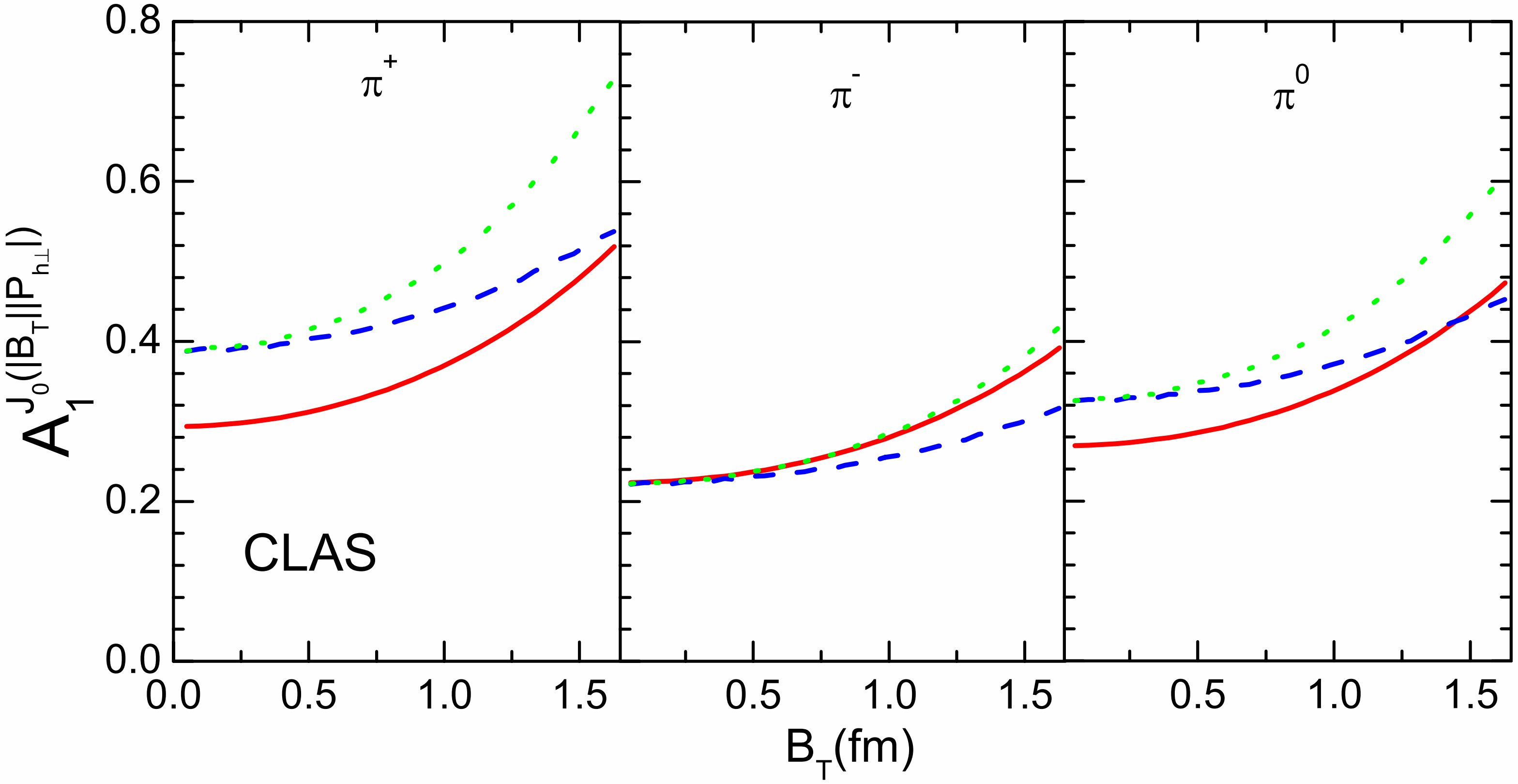



FIG. 5 (color online). The Bessel-weighted DSAs $A_1^{J_0(|\mathcal{B}_T||\boldsymbol{P}_{h\perp}|)}$ for $\pi^+$, $\pi^-$, and $\pi^0$ productions as functions of $\mathcal{B}_T$ at CLAS. The solid lines are from approach 2 of the light-cone diquark model, while the dashed line and the dotted lines are from the Gaussian ansatz for the TMD helicity distributions with $\langle p_T^2 \rangle_g^q = 0.17$ GeV and 0.10 GeV$^2$, respectively.

# Conclusions cont.

- Propose generalized Bessel Weights to study 3-D structure of the nucleon
- Bessel Weighting  solves problem of infinite contribution from large transverse  momentum that arise from using "conventional weighting
- Provides a regularization of infinite contributions at lg. transverse momentum when $\mathcal{B}_T^2$ is non-zero
- Soft, Hard CS, eliminated from weighted asymmetries, Sudakov dpnds coupling of *b* & *Q*
- Possible to compare observables at different scales.... could be useful for an EIC

# Conclusions cont.

- New experimental tool to study the TMD content at to the SIDIS  that minimize the transverse momentum model dependencies inherent in conventional extractions of TMDs.

- Impact for Lattice calculation of moments of TMDS,   B. Musch, Ph. Hagler, M. Engelhardt, J.W. Negele, A. Schafer  arXiv 2011

# Correlator w/explicit *spin orbit* correlations

$$\tilde{\Phi}^{[\gamma^+]}(x, \boldsymbol{b}_T) = \tilde{f}_1(x, \boldsymbol{b}_T^2) - i\, \epsilon_T^{\rho\sigma} b_{T\rho} S_{T\sigma}\, M\, \tilde{f}_{1T}^{\perp(1)}(x, \boldsymbol{b}_T^2)\,,$$

$$\tilde{\Phi}^{[\gamma^+\gamma^5]}(x, \boldsymbol{b}_T) = S_L\, \tilde{g}_{1L}(x, \boldsymbol{b}_T^2) + i\, \boldsymbol{b}_T \cdot \boldsymbol{S}_T M\, \tilde{g}_{1T}^{(1)}(x, \boldsymbol{b}_T^2)\,,$$

$$\begin{aligned}\tilde{\Phi}^{[i\sigma^{\alpha+}\gamma^5]}(x, \boldsymbol{b}_T) = {} & S_T^\alpha\, \tilde{h}_1(x, \boldsymbol{b}_T^2) + i\, S_L\, b_T^\alpha M\, \tilde{h}_{1L}^{\perp(1)}(x, \boldsymbol{b}_T^2) \\ & + \frac{1}{2}\left(b_T^\alpha b_T^\rho + \frac{1}{2}\, \boldsymbol{b}_T^2\, g_T^{\alpha\rho}\right) M^2\, S_{T\rho} \tilde{h}_{1T}^{\perp(2)}(x, \boldsymbol{b}_T^2) \\ & - i\, \epsilon_T^{\alpha\rho} b_{T\rho} M \tilde{h}_1^{\perp(1)}(x, \boldsymbol{b}_T^2)\,,\end{aligned}$$

N.B. $b_T$ Transverse sep. of quarks in correlator

# Large $b_T$

$$\frac{d\sigma}{dP_T^2} \propto \sum_{jj'} \mathcal{H}_{jj',\,\mathrm{SIDIS}}(\alpha_s(\mu), \mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T\cdot\boldsymbol{P}_T}\; \tilde{F}_{j/H_1}(x, b_T; \mu, \zeta_1)\, \tilde{D}_{H_2/j'}(z, b_T; \mu, \zeta_2)$$

$$+\; Y_{\mathrm{SIDIS}} \;+ \mathrm{P.S.C} \quad O(\Lambda/Q)^a$$

Practical issue: is that the ”TMD contribution” term is calculated in coordinate space and Fourier transformed back into momentum space

Calculations of *FT* term include non-perturbative behavior at large $b_T$

In the Fourier transforms that connect these calculations to cross sections, non-perturbative effects from large $b_T$ can migrate to unexpectedly large $P_T$, and perturbative effects from small $b_T$ can migrate to small $P_T$.

Must match these regions