Why does it make sense to divide viscosity η by entropy density s ?

An incomplete speculation

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My question

Talk of “perfect fluids” inevitably leads to discussion of how to judge that viscosity is “small”; what should it be scaled to?

The standard answer is entropy density, ie that the “natural” dimensionless quantity is η/s.

I have often asked why this should be the natural combination, but I’ve never gotten back a better answer than simply “They’re both intensive quantities with the same units.” That’s no unreasonable, but can we do better?

My speculation

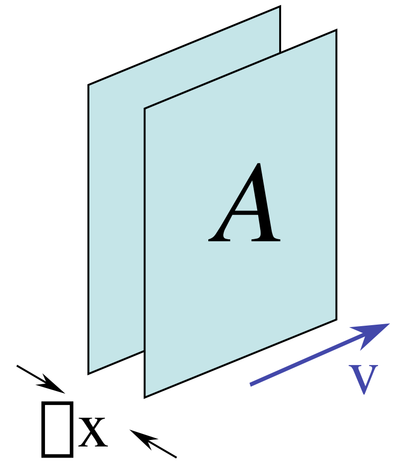
A feature of ideal hydrodynamics is that any ideal fluid's evolution is *isentropic*; another way of saying this is that such a system retains local thermal equilibrium *without* moving toward global thermal equilibrium.

Rather than describe the limit of ideal/perfect fluids in terms of mechanical quantities like viscosity and heat conductivity, it might make more sense in the big picture to say that the ideal/perfect fluid limit is when entropy generation is at a minimum.

I can make some sense of this by showing how viscosity is connected directly with entropy generation in the simplest possible example.

The definition of shear viscosity

The standard definition of shear viscosity is to imagine a fluid between two infinite parallel plates moving at a constant relative velocity. Shear viscosity η is the proportionality between the velocity gradient and the force required, per area, to keep the plates moving at constant velocity.



$$F = \eta \frac{v}{\Delta x} A = \eta \frac{\partial v}{\partial x} A$$

It requires work to maintain the velocity gradient:

$$\text{Power} \quad P = Fv = v\,\eta \frac{v}{\Delta x} A = \eta \left(\frac{\partial v}{\partial x}\right)^2 A\,\Delta x = \eta \left(\frac{\partial v}{\partial x}\right)^2 \times \text{Volume}$$

Creating entropy

Since the fluid's velocity profile is not changing, all the power must go into heating the fluid, ie into internal energy; we can re-cast this as entropy creation:

$$P = \frac{dE}{dt} = \frac{dE}{dS}\frac{dS}{dt} = T\frac{dS}{dt} = \eta\left(\frac{\partial v}{\partial x}\right)^2 \times \mathrm{Volume}$$

With s=S/V we divide by volume and then again by s

$$\frac{1}{s}\frac{ds}{dt} = \frac{\eta}{s}\left(\frac{\partial v}{\partial x}\right)^2 \frac{1}{T}$$

We see that η/s appears naturally as proportional to the fractional rate of increase in local entropy density.

Good news, but lingering questions

We've easily (almost trivially) shown a first-order relationship between two kinds of non-ideal fluid behavior: the mechanical (shear viscosity) and the more big-picture theoretical (entropy creation). But this is clearly not the whole story:

The scaled viscosity η/s is not *equal* to entropy creation *per se*, it's only *proportional* to entropy creation. What are those other factors telling us?

There are no c's in our equations, loosely suggesting that this relationship between local quantities should be just as valid for relativisitic fluids. But how would relativity enter?

It's tempting to note that the other factors $(dv/dx)^2(1/T)$ in the RHS of the final equation have the dimensions of 1/time^2-energy, and so we can write them as $1/(\tau_0 \hbar$bar$)$ for some time scale τ_0. What does that scale mean? And, how would quantum mechanics be entering this otherwise classical picture?