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Some Probability Distribution Functions and Their Relation Originally part of T. Abbott, *et al.*, Phys. Rev. C52, 2663 (1995)

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I. REVIEW OF PROBABILITY DISTRIBUTIONS

A review of some familiar distributions from probability and statistics is presented here, taken from the original discussion in "Multiplicity distributions from central collisions of ^{16}O+Cu at 14.6A GeV/c and intermittency," T. Abbott, *et al.*, Phys. Rev. C**52**, 2663 (1995), which was deleted by the referee who said that it belonged in a textbook.

A. Binomial Distribution

The binomial distribution is the result of repeated independent trials, each with the same two possible outcomes: success, with probability p, and failure, with probability $q = 1 - p$. The probabilities must remain the same for all the independent trials. The probability for m successes on n trials is:

$$P(m)\big|_n = \frac{n!}{m!(n-m)!}\, p^m (1-p)^{n-m} \tag{1}$$

and the mean, standard deviation and $F_2 - 1$ of the distribution are

$$\mu \equiv < m >= np \qquad \sigma = \sqrt{np(1-p)} \qquad \frac{\sigma^2}{\mu^2} = \frac{1}{\mu} - \frac{1}{n} \qquad F_2 - 1 = -\frac{1}{n} \qquad . \tag{2}$$

A distinguishing feature of the Binomial Distribution is that $F_2 - 1$ is negative.

B. Poisson Distribution

The Poisson Distribution is the limit of the Binomial Distribution for a large number of independent trials, n, with small probability of success, p, such that the expectation value of the number of successes per trial, $\mu = np$, remains fixed:

$$P(m)\big|_\mu = \frac{\mu^m e^{-\mu}}{m!} \tag{3}$$

$$<m>= \mu \qquad \sigma = \sqrt{\mu} \qquad \frac{\sigma^2}{\mu^2} = \frac{1}{\mu} \qquad . \tag{4}$$

For the Poisson distribution, $F_2 - 1 = 0$, and, indeed, for all orders of normalized factorial moments: $F_q - 1 = 0$. The Poisson distribution is intimately linked to the exponential law of Radioactive Decay of Nucleii[72, 73], the time distribution of nuclear disintegration counts, giving rise to the common usage of the term[72] "statistical fluctuations" to describe the Poisson statistics of such counts. The only assumptions required are the independence and constant probability of all trials—in other words, for a sample of radioactive material[72], the decay probability is the same for all atoms, is proportional to the time interval (for small intervals), is independent of time and is independent of the decay of other atoms.

C. Negative Binomial Distribution

The Negative Binomial Distribution of an integer m is defined as

$$P(m) = \frac{(m+k-1)!}{m!(k-1)!} \, \frac{(\frac{\mu}{k})^m}{(1+\frac{\mu}{k})^{m+k}} \tag{5}$$

where $P(m)$ is normalized for $0 \leq m \leq \infty$, $\mu \equiv < m >$, and some higher moments are:

$$\sigma = \sqrt{\mu(1+\frac{\mu}{k})} \qquad \frac{\sigma^2}{\mu^2} = \frac{1}{\mu} + \frac{1}{k} \qquad F_2 = 1 + \frac{1}{k} \tag{6}$$

The normalized factorial moments (F_q) and normalized factorial Cumulants (K_q)[33, 34] of the NBD are particularly simple:

$$F_q = F_{(q-1)}(1+\frac{q-1}{k}) \qquad\qquad K_q = \frac{(q-1)!}{k^{q-1}} \qquad . \tag{7}$$

The Binomial Distribution gives the probability of m successes on n repeated independent trials, each with the same probability p of success and $1-p$ of failure, while the Negative Binomial Distribution gives the probability that the k'th success occurs on the n'th trial, where $m = n - k$ represents the number of trials more than the desired number of successes. Alternatively, the NBD is the distribution of the number of trials more than the number of successes, $m = n - k$, for a fixed number of successes, k:

$$P(n)|_k = P(m)|_k = \frac{(m+k-1)!}{m!(k-1)!} \, p^k \, (1-p)^m \tag{8}$$

and $P(m)$ is normalized for $0 \leq m \leq \infty$. This goes to the standard form (Eq. 5) with the substitution

$$<m>= \mu \quad p = \frac{1}{1+\frac{\mu}{k}} \quad 1-p = \frac{\frac{\mu}{k}}{1+\frac{\mu}{k}} \tag{9}$$

The NBD, with an additional parameter k compared to a Poisson distribution, becomes Poisson in the limit $k \rightarrow \infty$ and Binomial for k equal to a negative integer (hence the name). The extra parameter has made the NBD useful to mathematical statisticians as a test for whether a distribution is Poisson—more precisely as a "test for independence in rare events."[35] The test for a Poisson distribution consists of determining whether the NBD parameter $1/k$ is consistent with zero to within its error $s_{\frac{1}{k}}$, which is given[35] as:

$$s_{\frac{1}{k}} = \frac{s_k}{k^2} = \frac{1}{\mu}\sqrt{\frac{2}{N}} \tag{10}$$

where N is the total number of events. For statisticians, the NBD represents the first departure from a Poisson Law. Physicists are more likely to describe the NBD as Bose-Einstein ($k = 1$) or Generalized Bose Einstein $k \neq 1$ distributions[6].

D. Gamma Distribution

In distinction to the previous distributions which are defined for integers, the Gamma distribution represents the probability density for a continuous variable x and has a parameter p (which is not to be confused with the symbol for probability used above):

$$f(x) = \frac{b}{\Gamma(p)}(bx)^{p-1}e^{-bx} \tag{11}$$

where

$$p > 0, \quad b > 0, \quad 0 \leq x \leq \infty$$

$\Gamma(p) = (p-1)!$ if p is an integer, and $f(x)$ is normalized. The first few moments of the distribution are

$$\mu \equiv < x > = \frac{p}{b} \qquad \sigma = \frac{\sqrt{p}}{b} \qquad \frac{\sigma^2}{\mu^2} = \frac{1}{p} \qquad F_2 - 1 = \frac{(1-b)}{p} \qquad . \tag{12}$$

The Gamma distribution has an important property under convolution. Define the n-fold convolution of a distribution with itself as :

$$f_n(x) = \int_0^x dy\, f(y)\, f_{n-1}(x-y) \quad ; \tag{13}$$

then for a Gamma distribution (Eq. 11), the n-fold convolution is simply given by the function

$$f_n(x) = \frac{b}{\Gamma(np)}(bx)^{np-1}e^{-bx} \tag{14}$$

i.e. $p \to np$ and b remains unchanged. Notice that the mean μ_n and standard deviation σ_n of the n-fold convolution obey the familiar rule

$$\mu_n = n\mu \qquad \sigma_n = \sigma\sqrt{n} \quad . \tag{15}$$

II. SUMS OF INDEPENDENT AND CORRELATED RANDOM VARIABLES

In mathematical statistics[74], the probability distribution of a random variable S_n, which is itself the sum of n independent random variables with a common distribution $f(x)$:

$$S_n = x_1 + x_2 + \cdots + x_n \tag{16}$$

is just $f_n(x)$, the n-fold convolution of the distribution $f(x)$. This explains why convolutions, and the Gamma Distribution with its simple behavior, are so useful for E_T and multipicity, which are variables of the form of Eq. 16. There is a particularly interesting and direct application of the Gamma distribution to the time interval between every nth count of radioactive decay, where the probability is exponential for the time interval between counts[72, 73, 75]. Since an exponential is just a Gamma distribution (Eq. 11) with $p = 1$, the distribution for the time x between n counts is just given by Eq. 14, with $p = 1$, and $b = \lambda$, the normalized probability of decay per unit time.

Another complementary case is that of a random variable Z_n, which is the sum of n random variables with distribution $f(x)$—which are themselves 100% correlated—for example:

$$Z_n = x + x + \cdots + x = nx \qquad . \tag{17}$$

This is just a scale transformation. The behavior of the mean and the standard deviation for a scale transformation is $\mu \rightarrow n\mu$, $\sigma \rightarrow n\sigma$, which is quite different than the more familiar behavior of the standard deviation under convolution (Eq. 15). The result of the scale transformation $x \rightarrow nx$ for a Gamma distribution (Eq. 11) is simply $b \rightarrow b/n$, with p remaining unchanged. The most interesting example of a scale transformation for a Gamma distribution is scaling by the mean value, $\mu =<x>$, or $x \rightarrow x/\mu$, $p \rightarrow p$, $b \rightarrow \mu b = p$, with the result:

$$\psi(z) =<x> f(x) = \frac{p}{\Gamma(p)}(pz)^{p-1}e^{-pz} \qquad \text{where} \qquad z = \frac{x}{<x>} \tag{18}$$

and $\psi(z)$ is normalized for $0 \leq z \leq \infty$. Thus the Gamma distribution has the property of "scaling in the mean", which means that the shape of the distribution Eq. 18 is determined only by the parameter p, independently of the mean value μ. This property does not hold in general and is not satisfied for Poisson or Negative Binomial distributions. In particle physics, "scaling in the mean" is usually called KNO scaling[76].

A. Further properties of the Negative Binomial Distribution

The Negative Binomial Distribution bears a strong relationship to the Gamma distribution, and becomes a Gamma distribution in the limit $\mu \gg k > 1$. In fact, many times, Gamma distributions are substituted for NBD to prove various theorems[77]. The convolution property of the Gamma distribution also holds for the NBD. The probability distribution of the sum of n independent variables, each distributed as an NBD with mean μ and parameter k, is the n-fold convolution of the distribution, which is an NBD with mean $n\mu$ and parameter nk, so that the ratio μ/k remains constant for the convolutions exactly like the Gamma distribution. Furthermore, the familiar rule for the mean and standard deviation (Eq. 15) is satisfied. It is convenient, in analogy to the Gamma distribution, to introduce the parameter

$$b \equiv \frac{k}{\mu} \quad \text{so that} \quad < m > \equiv \mu = \frac{k}{b} \quad , \tag{19}$$

and then to write the NBD, particularly for large k, as:

$$P(m) = \frac{1}{(1+\frac{1}{b})^k} \frac{k(k+1)\cdots(k+m-1)}{1\cdot 2\cdots m\ (b+1)^m} \quad . \tag{20}$$

The only important difference between between NBD and Gamma distributions is in the limit m or $x \to 0$: for $p > 1$ the limit is always zero for a Gamma distribution, whereas for the NBD it is always finite.

B. Relationship of the Binomial, Poisson, Gamma and Negative Binomial distributions

The history of the use of the Poisson distribution by statisticians includes the study of the number of accidental deaths by horse kicks in the Prussian army[35]. However, the Poisson distribution did not work for the case of factory accidents because different workers had different chances of having an accident. If the mean probability of an accident per worker for different workers is distributed according to a Gamma distribution (also known as Pearson Type III), then the NBD rather than the Poisson is the resultant distribution for all workers[35, 41, 78, 79].

In addition to the variation of a Poisson mean value leading to a NBD, any tendency of events to occur in groups instead of independently, spreads the variance and makes the distribution more like a NBD. For instance, in repeated binomial trials, there could be some correlation such that some of the outcomes represent more than one success. If the relative probabilities p_1 for one success, p_2 for 2 successes, p_n for n successes on a trial form the series, $p_n = p_1 a^{n-1}/n$, then the overall distribution is again Negative Binomial[35]. Thus it has been stated that[35] "the agreement of the data about deaths from kicks of a horse in the Prussian army may be taken to mean both (1) that nobody can be killed twice by the kick of a horse, (2) that the fact that one man has been so killed does not indicate an extra liability for others in the same unit to be. The agreement in the radioactivity law would mean that (1) the chances of disintegration of different atoms of the same radioactive substance are approximately equal, (2) the disintegration of one atom does not lead immediately to to the disintegration of another."

C. Compound distributions

A compound distribution results from the sum S_n (Eq. 16) of n independent random variables with a common distribution $f(x)$, when n is itself a random variable, independent of x. The two interesting examples for the present discussion concern the case where n is either Poisson or Negative Binomial and $f(x)$ is binomial ($x = 1$ for a success, with probability p; and $x = 0$ for a failure, with probability $q = 1 - p$). Thus, the probability $P(S_n = m)$ for a fixed n is given by Eq. 1, and n varies randomly according to a distribution. This compound distribution is easier to visualize if one defines $A \equiv S_n$ as the number of successes on n trials, and B as the number of failures on n trials, so that the random variable n is the sum: $n = A+B$. The random variable n is composed of two distinct sub-populations: A, on the interval p; and B, on the interval $1 - p$.

If the random variable n (the number of trials) is Poisson distributed with mean μ, then the distribution of A, the number of successes, is also Poisson with mean $<A>= p\mu$, where p is the Binomial probability for success on a single trial. More importantly, the distributions of the two sub-populations A and B on the sub-intervals p and $1 - p$ are statistically independent. Again the Poisson distribution forms the "intuition" that the Binomial division of a population gives two statistically independent (Poisson) sub-populations which can be summed to obtain the original population.

For the case where n is Negative Binomial with mean μ and parameter k, the compound distribution A is also NBD[41, 63] with mean $p\mu$ and the same parameter k[81]. However, for the NBD, the distributions on the two sub-intervals are *not* statistically independent—the distribution on one sub-interval depends explicitly on the result on the other sub-interval. In other words, if $z = A+B$, where the probability for z is NBD, and at fixed z the probability for A and B is Binomial, then "the average of A is a linear function of B and vice-versa."[7] A corollary of this result is that if A and B are independent random variables with a common NBD, and $z = A + B$, then the probability of A and B for fixed z is *not* Binomial[82].

This characteristic property of a compound Negative Binomial Distribution has important physical consequences—forward-backward (long-range) correlations[5, 6, 80, 83]. Conversely, the search for a functional form for multiplicity distributions that supported the observed[5, 80] forward-backward correlations (where the mean backward multiplicity is linearly proportional to the forward multiplicity) led to the Negative Binomial Distribution[7, 84].

III. FORMALISM OF MULTIPARTICLE CORRELATIONS

The "inclusive probability density" per interaction of observing a particle at rapidity y is:

$$\rho_1(y) = \frac{1}{\sigma}\frac{d\sigma}{dy} = \frac{dn}{dy} \tag{21}$$

where σ is the interaction cross section. The joint probability density for a particle at y_1 and another at y_2 is:

$$\rho_2(y_1, y_2) = \frac{1}{\sigma}\frac{d^2\sigma}{dy_1 dy_2} \tag{22}$$

and for q particles at $y_1, y_2 \ldots y_q$

$$\rho_q(y_1, \ldots y_q) = \frac{1}{\sigma}\frac{d^q\sigma}{dy_1 \cdots dy_q} \quad . \tag{23}$$

If there is no correlation, then the emission of the particles is statistically independent, and

$$\rho_2(y_1, y_2) = \rho_1(y_1)\rho_1(y_2) \tag{24}$$

$$\rho_q(y_1, \ldots y_q) = \rho_1(y_1)\rho_1(y_2) \cdots \rho_1(y_q) \qquad . \tag{25}$$

Mueller[33] introduced a series of moments and correlation functions to describe multiparticle correlations. The Mueller correlation functions, C_q, are the q-particle rapidity densities,

ρ_q, with all combination of lower order q-particle correlations subtracted out, so that $C_q = 0$ for statistically independent emission. The most straightforward Mueller correlation function is

$$C_2(y_1, y_2) = \rho_2(y_1, y_2) - \rho_1(y_1)\rho_1(y_2) \quad , \tag{26}$$

which obviously vanishes for the case of no correlation.

The integrals of the inclusive q-particle densities, $\rho_q(y_1, \dots y_q)$, on any interval are the unnormalized factorial moments on that interval,

$$\int dy_1 \int dy_2 \; \rho_2(y_1, y_2) =< n(n-1) >=<n>^2 \; F_2 \tag{27}$$

$$\int dy_1 \cdots dy_q \; \rho_q(y_1, \dots y_q) =< n(n-1) \dots (n-q+1) >=<n>^q \; F_q \quad ; \tag{28}$$

while the integrals of the Mueller correlation functions are the Mueller moments, or unnormalized factorial Cumulants[33, 34, 41], $f_q \equiv <n>^q \; K_q$,

$$\int dy_1 \cdots \int dy_q \, C_q(y_1, \dots y_q) = f_q =<n>^q \; K_q \quad . \tag{29}$$

The Muller moments are just the unnormalized factorial moments with all q-particle combinations of lower order moments subtracted:

$$\begin{aligned} f_2 &= <n>^2 \; F_2 - f_1^2 \\ f_3 &= <n>^3 \; F_3 - (f_1^3 + 3 f_1 f_2) \\ f_4 &= <n>^4 \; F_4 - (f_1^4 + 6 f_1^2 f_2 + 3 f_2^2 + 4 f_1 f_3) \\ &\dots \end{aligned} \tag{30}$$

where $f_1 =< n >$. The Mueller correlation functions C_q can be read from Eqs. 28–30. The Mueller moments, f_q, and the normalized factorial Cumulants, $K_q = f_q / < n >^q$, are zero if there is no direct q-particle correlation. However, as the Mueller moments (which are unnormalized) go to zero trivially, in the limit $<n>\to 0$, the normalized factorial Cumulants must be used to test for direct q-particle correlations in this limit[71].

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[81] Here again, any clustering will cause k to vary. See W. Q. Chao, T. C. Meng, and J. C. Pan, *Phys. Lett.* **B176**, 211 (1986).

[82] The distribution can be found in M. J. Tannenbaum, E-802-MEM-58, internal memorandum, Brookhaven National Laboratory (1993).

[83] W. A. Zajc, *Phys. Lett.* **B175**, 219 (1986).

[84] Previously, the search for multiplicity distributions that obeyed KNO scaling[76] had led to Gamma distributions. See P. Slattery, *Phys. Rev.* **D7**, 2073 (1973).