Spatially Non-Flat Cosmologies

Intro Cosmology Short Course
Lecture 4

Paul Stankus, ORNL

Recall from Lecture 2:

Robertson-Walker Metric

$$d\tau^2 = dt^2 - [a(t)]^2 d\chi^2/c^2$$

1. The χ coordinate system is uniform (not necessarily Euclidean)

$$d\tau^2 = dt^2 - [a(t)]^2 \left[\sum_{ij} d\chi^i \underbrace{\gamma_{ij}^{(3)}(\chi^1,\chi^2,\chi^3)}_{\substack{\text{Space-space} \\ \text{Metric in 3D}}} d\chi^j \right]$$

General spatial metric

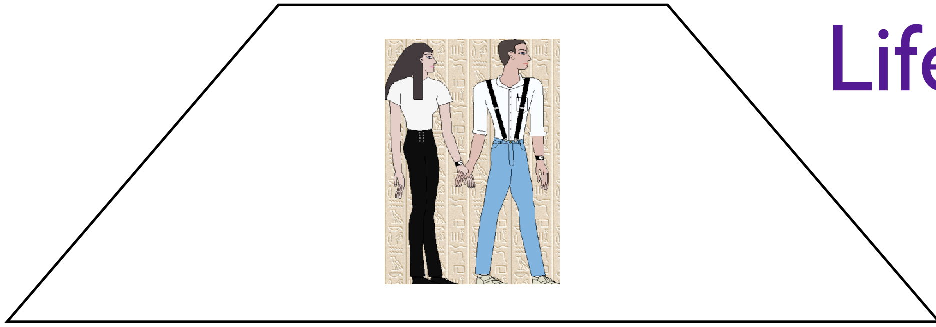
$$d\tau^2 = dt^2 - [a(t)]^2 \left[(d\chi^1)^2/c^2 + (d\chi^2)^2/c^2 + (d\chi^3)^2/c^2 \right]$$

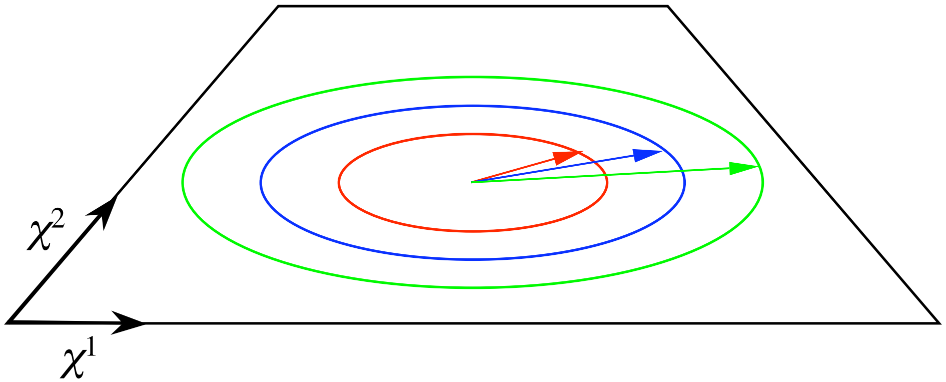
Euclidean case

What other isotropic, uniform 3-D spaces are possible?

Life in a 2-D plane

How do you test if your space is flat/Euclidean?



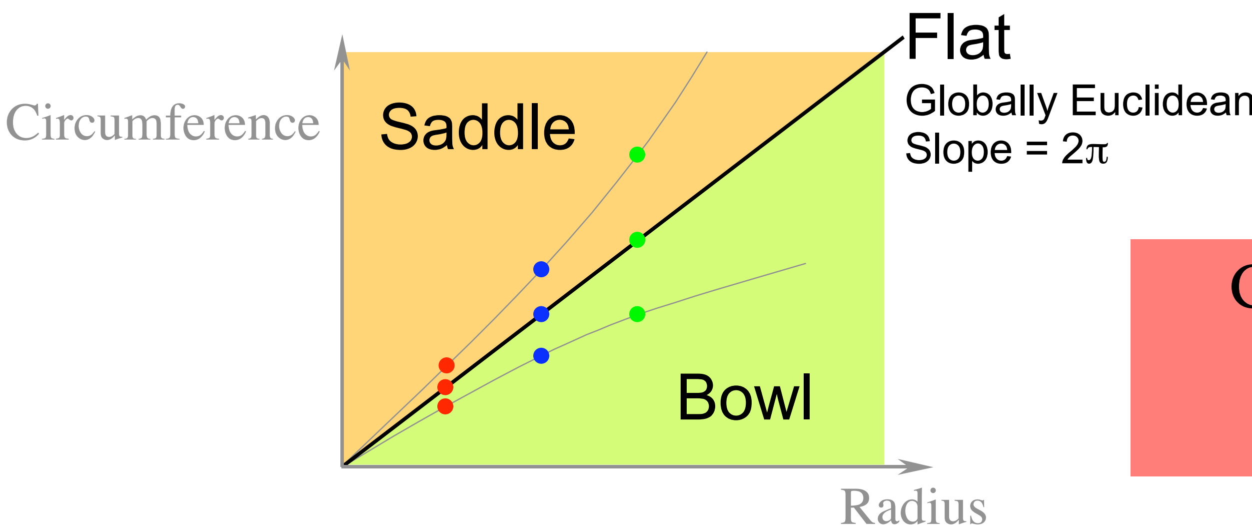


Assume space is locally Euclidean

Straight lines are well-defined

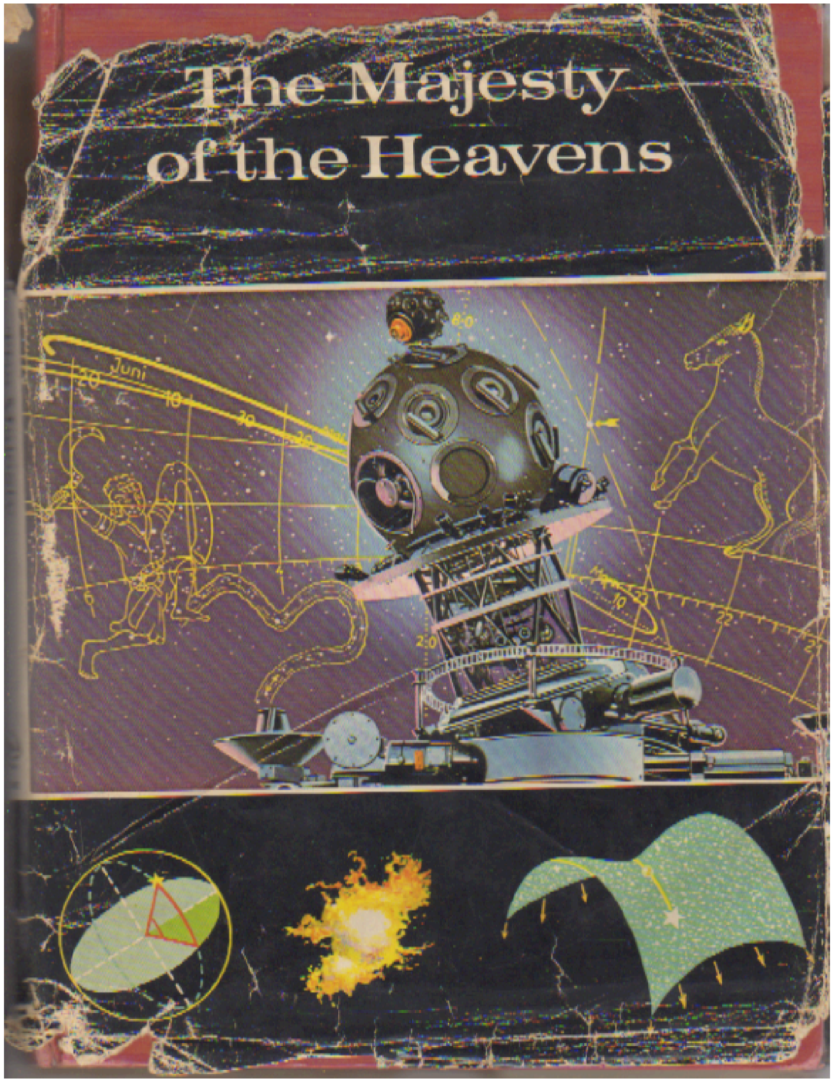
Circles can be constructed and their radii and circumferences measured

$$\text{Length} = \int_{\text{Path}} ds$$



Characterize by

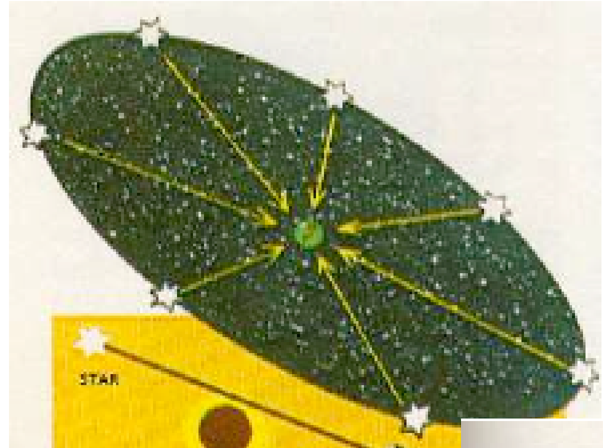
$$C(r)$$



My childhood astronomy book

Part of the "Foundations of Science" series

(Greystone, MCMLXVI)

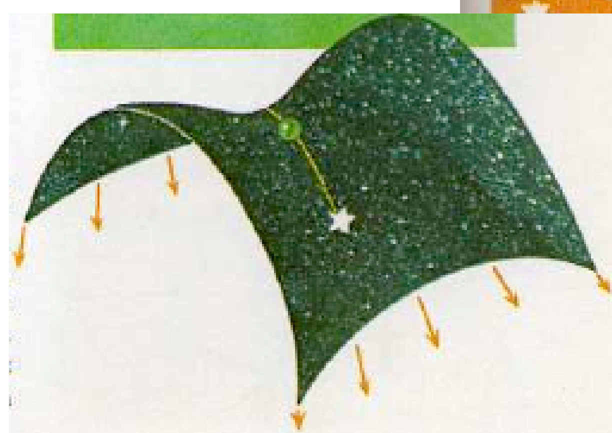


Flat

$C(r) = 2\pi r$

Bowl or Sphere

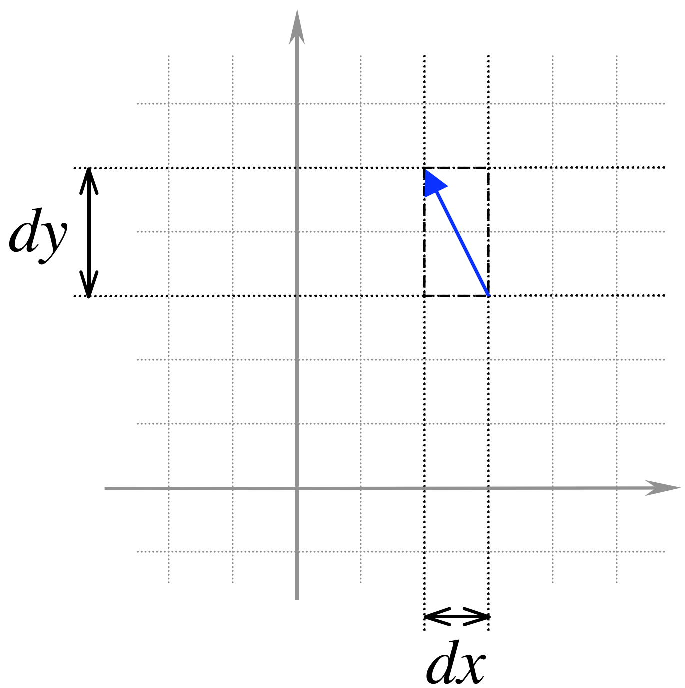
$C(r) < 2\pi r$

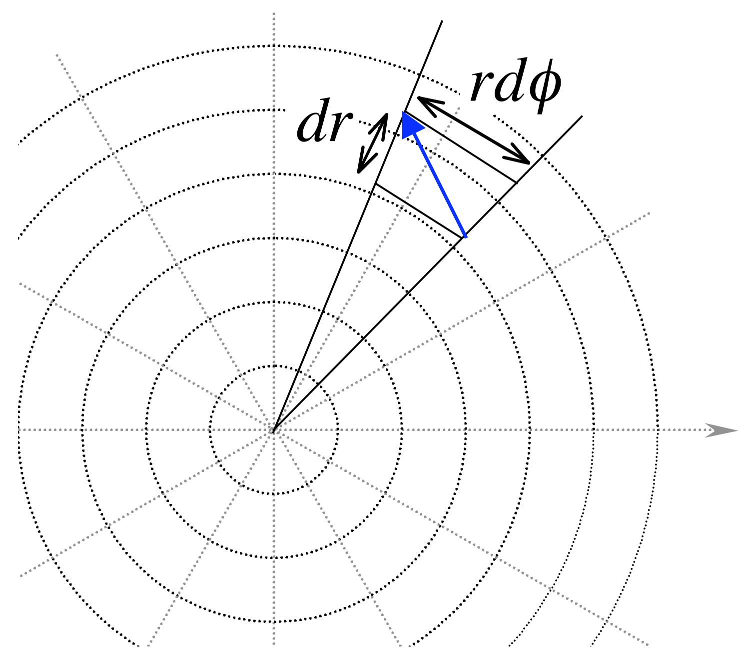


Saddle

$C(r) > 2\pi r$

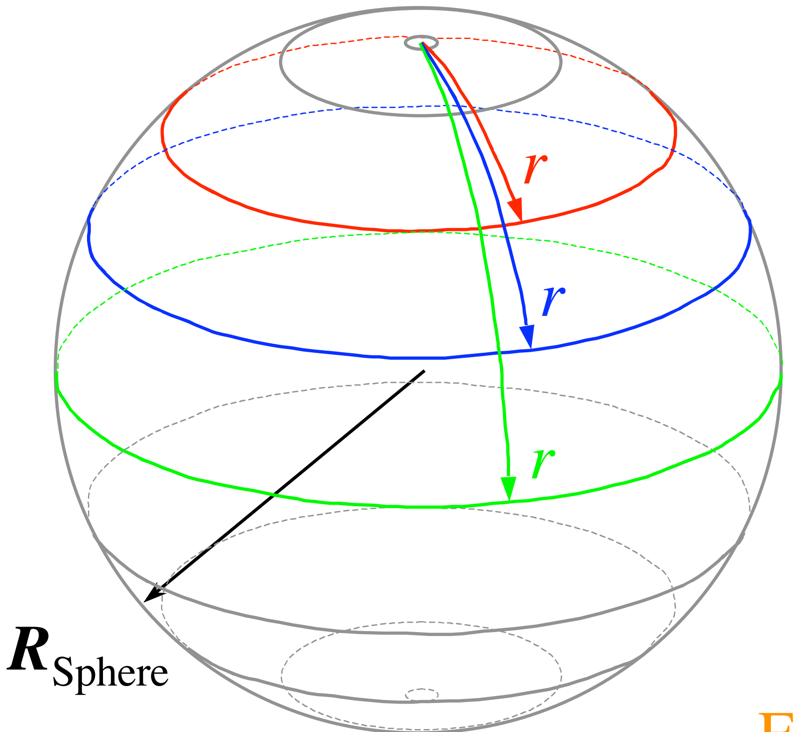
Metrics in flat 2-D space

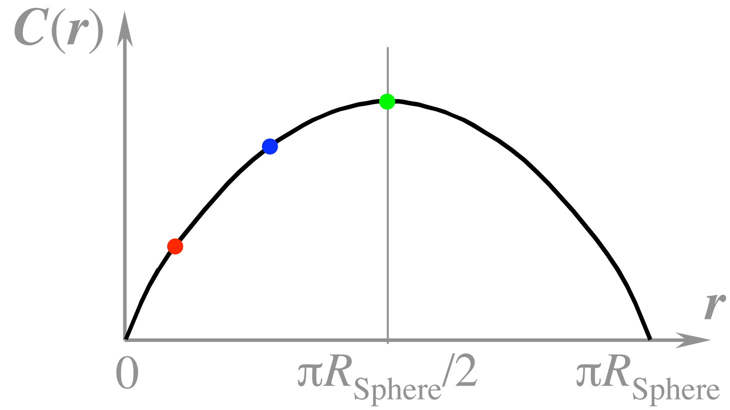




$$ds^2 = dx^2 + dy^2$$

$$ds^2 = dr^2 + r^2 d\phi^2$$



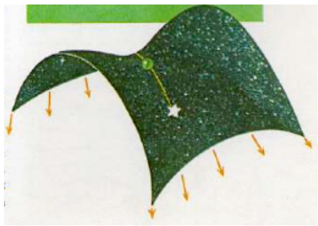


$$C(r) = 2\pi R_{\text{Sphere}} \sin(r/R_{\text{Sphere}})$$
$$\Rightarrow 2\pi r \quad \text{as} \quad r << R_{\text{Sphere}}$$

Euclid in 2-D $\quad ds^2 = dr^2 + r^2 d\phi^2$

Distance Metric in 2-D on Surface of a Sphere

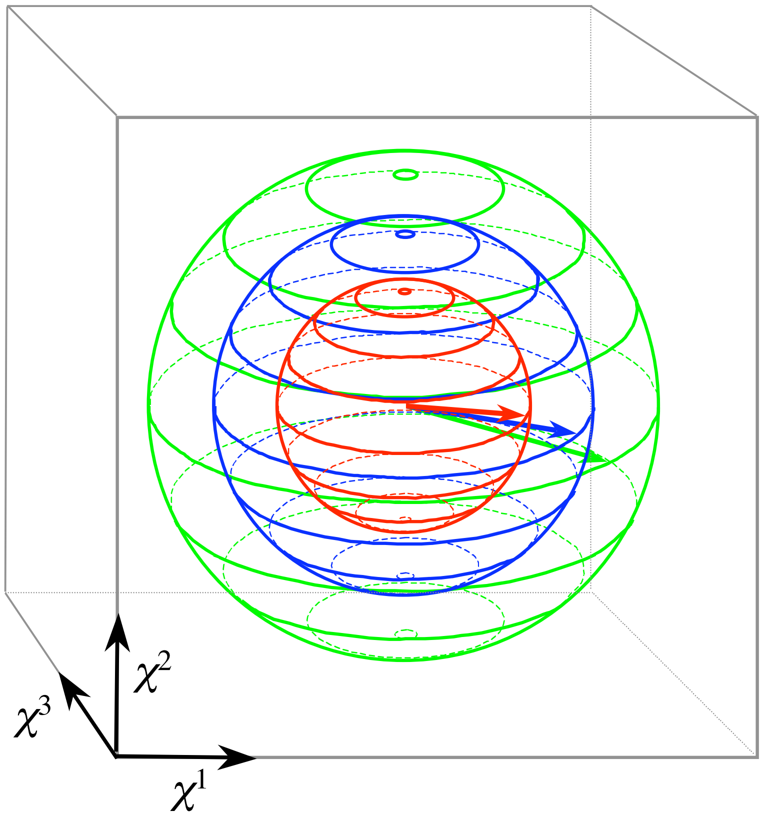
$$ds^2 = dr^2 + \left(C(r)/2\pi\right)^2 d\phi^2$$
$$= dr^2 + \left(R_{\text{Sphere}} \sin(r/R_{\text{Sphere}})\right)^2 d\phi^2$$



Generalization to Saddle Surface

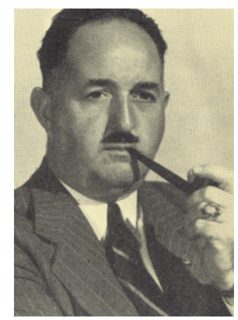
Let $R_{\text{Sphere}}{}^2 \rightarrow -R_{\text{Saddle}}{}^2$, $R_{\text{Sphere}} \rightarrow -iR_{\text{Saddle}}$

$$ds^2 = dr^2 + \left(R_{\text{Saddle}} \sinh(r/R_{\text{Saddle}})\right)^2 d\phi^2$$



$A < 4\pi\chi^2$	$A = 4\pi\chi^2$	$A > 4\pi\chi^2$
Hyper-sphere "Closed"	Euclidean "Flat"	Hyper-saddle "Open"
$k \equiv 1/R_{\text{Hyper-Sphere}}^2$	$k = 0$	$k \equiv -1/R_{\text{Hyper-Saddle}}^2$

Robertson-Walker
Metric in 3+1D



$$d\tau^2 = \begin{cases} dt^2 - [a(t)]^2 \left(d\chi^2 + |k|^{-1} \sin^2\left(\sqrt{|k|}\chi\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) \Big/ c^2 & \text{Closed} \\ dt^2 - [a(t)]^2 \left(d\chi^2 + \chi^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) \Big/ c^2 & \text{Flat} \\ dt^2 - [a(t)]^2 \left(d\chi^2 + |k|^{-1} \sinh^2\left(\sqrt{|k|}\chi\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) \Big/ c^2 & \text{Open} \end{cases}$$

Common Alternate Notation

Our notation (for k>0)

$$d\tau^2 = dt^2 - [a(t)]^2 \left(d\chi^2 + k^{-1} \sin^2\left(\sqrt{k}\,\chi\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) \Big/ c^2$$

$a(t)$ Dimensionless χ Length $k \sim 1/R_0^2$ $1/\text{Length}^2$

Older notation (e.g. Weinberg 14.2.1)

$$d\tau^2 = dt^2 - [R(t)]^2 \left(\frac{dr^2}{1 - kr^2} + r^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) \Big/ c^2$$

$R(t)$ Length $r \sim \chi/R_0$ Dimensionless k = -1,0,+1 Dimensionless

Spatial Curvature and Dynamics

The Friedmann Equation

$\Lambda=0$

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 \equiv H^2(t) = \frac{8\pi G_N \rho(t)}{3c^2} - \frac{k c^2}{\left(a(t)\right)^2}$$

GR

Define $\rho_{Crit}(t) \equiv \dfrac{3c^2 H^2(t)}{8\pi G_N}$

$$\frac{8\pi G_N}{3c^2}\left(\rho(t) - \rho_{Crit}(t)\right) = \frac{k c^2}{\left(a(t)\right)^2}$$

Define $\Omega_{M+R}(t) \equiv \dfrac{\rho_{M+R}(t)}{\rho_{Crit}(t)}$

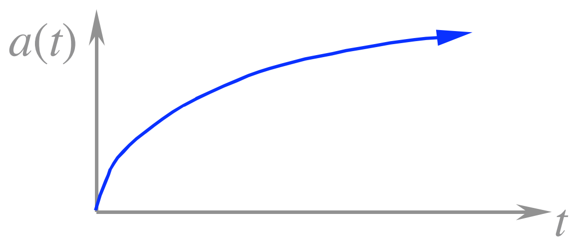
$$\Omega_{M+R}(t) - 1 = \frac{k c^2}{\left(H(t)\, a(t)\right)^2} = \frac{k c^2}{\left(\dot{a}(t)\right)^2}$$

Three Matter-Filled Universes

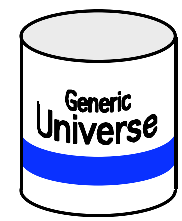
Assume all ρ in matter, then $\rho(t)=\rho_0/a(t)^3$

$$\left(\dot{a}(t)\right)^2 = \frac{8\pi G_\mathrm{N}}{3c^2}\frac{\rho_0}{a(t)} - kc^2$$

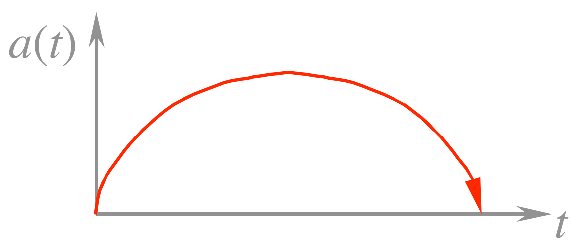
k=0
Flat



Invert directly
$a(t) \propto t^{2/3}$, $\dot{a}(t) \propto t^{-1/3}$

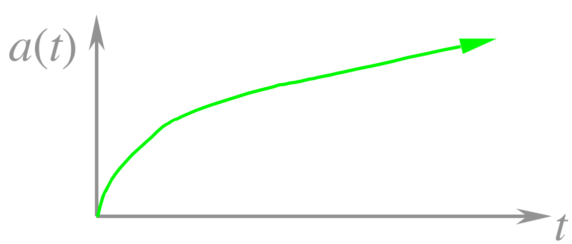


k>0
Closed



$\dot{a}(t)$ must hit 0
Recollapse!

k<0
Open



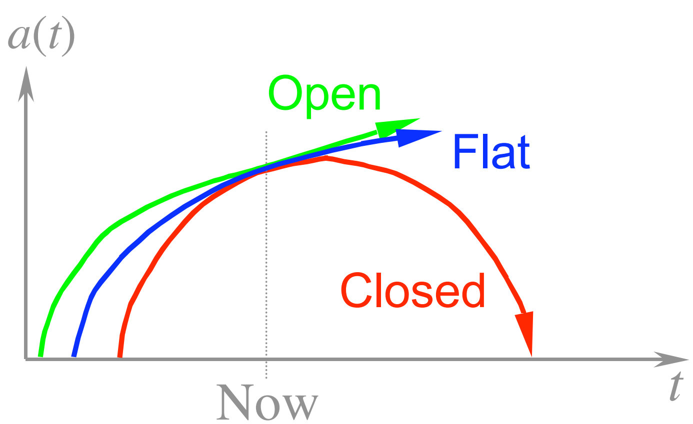
$\dot{a}(t) \rightarrow$ Constant

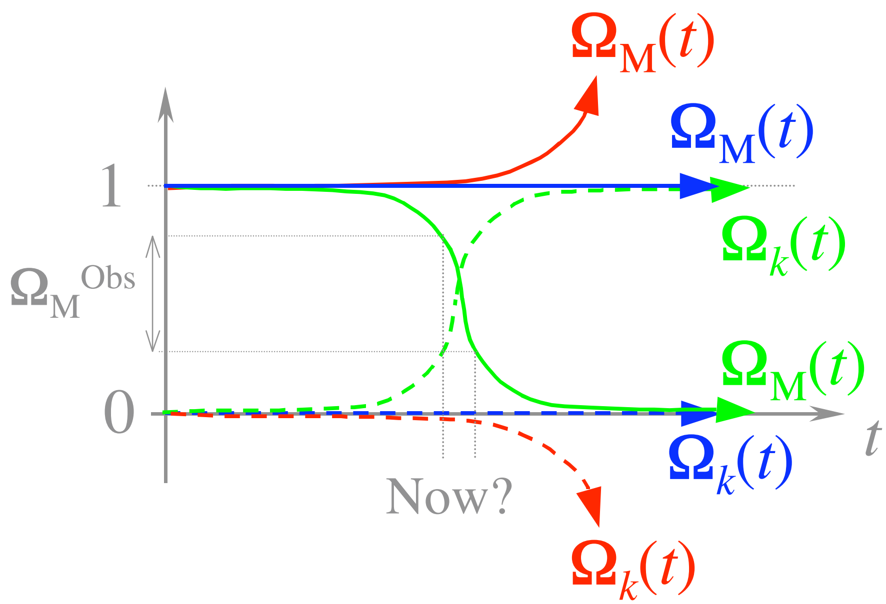
Do we live at a special time (1)?

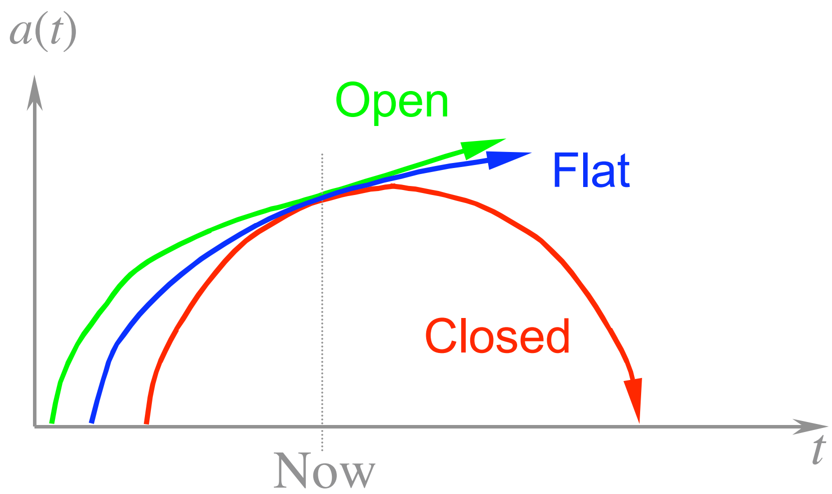
Define $\Omega_k(t) \equiv -\dfrac{kc^2}{\left(H(t)\,a(t)\right)^2}$

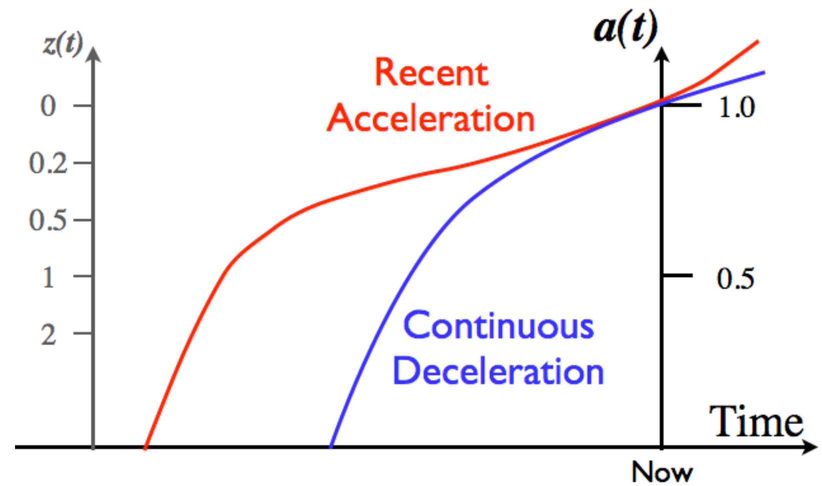
$$\Omega_{\mathrm{M+R}}(t) + \Omega_k(t) = 1$$

$\Lambda=0$

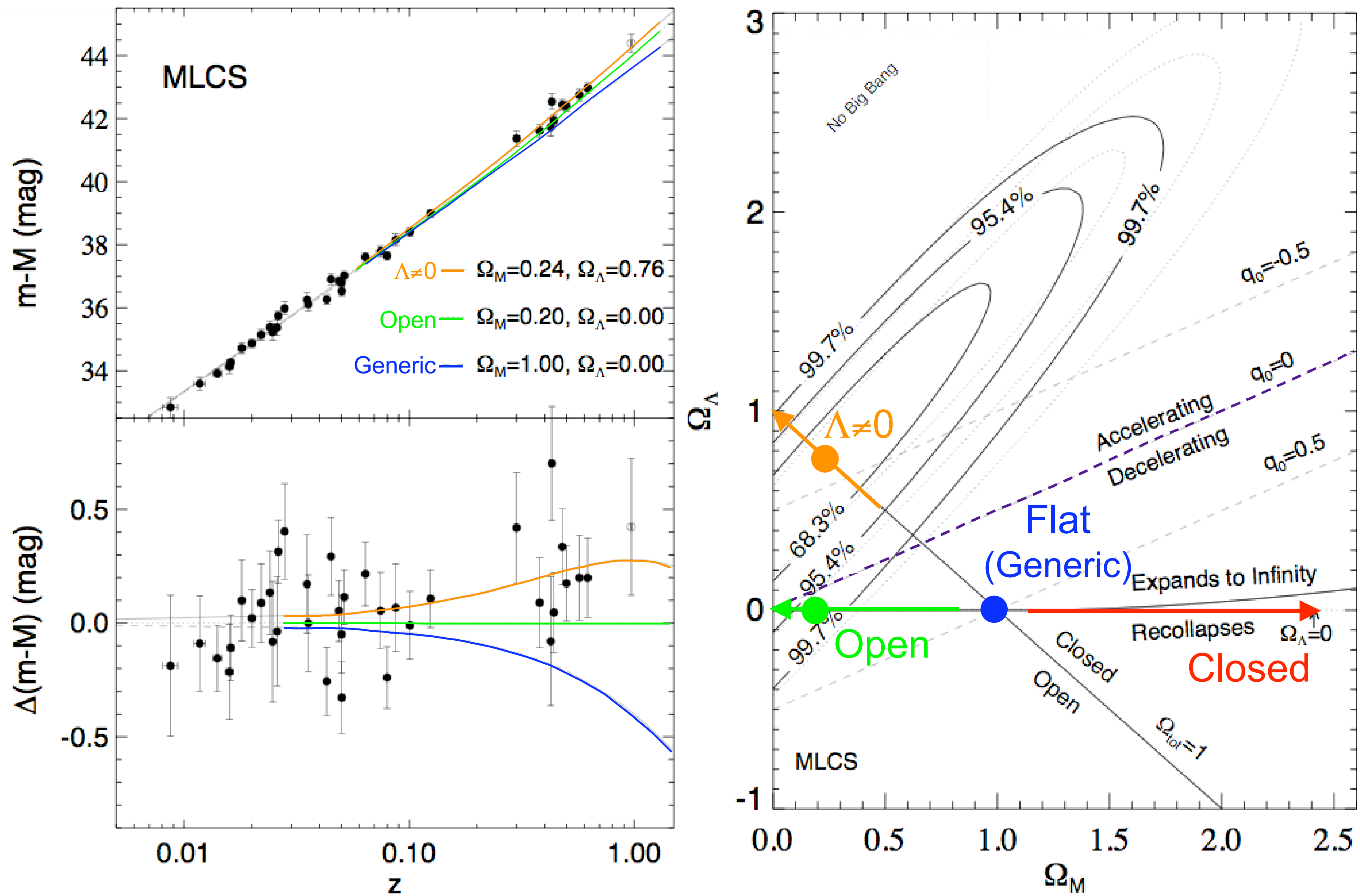








	Then (1930-1998)	**Now** (1998-2007)
Working belief	$\Lambda=0$	$k\approx0$
Big question	Open, closed or flat?	Accelerating or decelerating?
Key parameters	$\mathbf{\Omega}_{\mathrm{M}}\,,\ \mathbf{\Omega}_{k}$	$\mathbf{\Lambda}$, $d\mathbf{\Lambda}/dt$; w, dw/dt
Coincidence?	$\mathbf{\Omega}_{\mathrm{M}} \sim \mathbf{\Omega}_{k}$	$\mathbf{\Omega}_{\mathrm{M}} \sim \mathbf{\Omega}_{\Lambda}$



MLCS
m-M (mag)
Λ≠0 — Ω_M=0.24, Ω_Λ=0.76
Open — Ω_M=0.20, Ω_Λ=0.00
Generic — Ω_M=1.00, Ω_Λ=0.00
Δ(m-M) (mag)
z
No Big Bang
95.4%
99.7%
68.3%
q_0=-0.5
q_0=0
q_0=0.5
Accelerating
Decelerating
Λ≠0
Flat
(Generic)
Expands to Infinity
Recollapses
Ω_Λ=0
Open
Closed
Ω_{tot}=1
MLCS
Ω_Λ
Ω_M

Points to take home

- Uniform but non-Euclidean spaces: “bowls and saddles”
- Bowl: $C(r)<2\pi r$, $A(r)<4\pi r^2$ Saddle: $C(\mathrm{r})>2\pi r$, $A(r)>4\pi r^2$
- Generalize Robertson-Walker metric
- Friedmann equation connects density, expansion, curvature
- Matter-dominated: open, flat, or closed? expansion or recollapse?
- Do we live at a special time? Part 1