Spherical Harmonic

The spherical harmonics $Y_l^m(\theta, \phi)$ are the angular portion of the solution to Laplace's equation in spherical coordinates where azimuthal symmetry is not present. Some care must be taken in identifying the notational convention being used. In this entry, θ is taken as the polar (colatitudinal) coordinate with $\theta \in [0, \pi]$, and ϕ as the azimuthal (longitudinal) coordinate with $\phi \in [0, 2\pi)$. This is the convention normally used in physics, as described by Arfken (1985) and *Mathematica* (in mathematical literature, θ usually denotes the longitudinal coordinate and ϕ the colatitudinal coordinate). Spherical harmonics are implemented in *Mathematica* as `SphericalHarmonicY`[*l*, *m*, *theta*, *phi*].

Spherical harmonics satisfy the spherical harmonic differential equation, which is given by the angular part of Laplace's equation in spherical coordinates. Writing $F = \Phi(\phi)\Theta(\theta)$ in this equation gives

$$\frac{\Phi(\phi)}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{\Theta(\theta)}{\sin^2\theta}\frac{d^2\Phi(\phi)}{d\phi^2} + l(l+1)\Theta(\theta)\Phi(\phi) = 0. \tag{1}$$

Multiplying by $\sin^2\theta/(\Theta\Phi)$ gives

$$\left[\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta\right] + \frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = 0. \tag{2}$$

Using separation of variables by equating the ϕ-dependent portion to a constant gives

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} = -m^2, \tag{3}$$

which has solutions

$$\Phi(\phi) = Ae^{-im\phi} + Be^{im\phi}. \tag{4}$$

Plugging in (3) into (2) gives the equation for the θ-dependent portion, whose solution is

$$\Theta(\theta) = P_l^m(\cos\theta), \tag{5}$$

where $m = -l$, $-(l-1)$, ..., 0, ..., $l-1$, *l* and $P_l^m(z)$ is an associated Legendre polynomial. The spherical harmonics are then defined by combining $\Phi(\phi)$ and $\Theta(\theta)$,

$$Y_l^m(\theta, \phi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}, \tag{6}$$

where the normalization is chosen such that

$$\int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta,\phi)\bar{Y}_{l'}^{m'}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \int_0^{2\pi}\int_{-1}^{1} Y_l^m(\theta,\phi)\bar{Y}_{l'}^{m'}(\theta,\phi)\,d(\cos\theta)\,d\phi \tag{7}$$

$$= \delta_{mm'}\delta_{ll'} \tag{8}$$

(Arfken 1985, p. 681). Here, $\bar{z}$ denotes the complex conjugate and δ_{mn} is the Kronecker delta.

Sometimes (e.g., Arfken 1985), the Condon-Shortley phase $(-1)^m$ is prepended to the definition of the spherical harmonics.

The spherical harmonics are sometimes separated into their real and imaginary parts,

$$Y_l^{ms}(\theta,\phi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\sin(m\phi) \tag{9}$$

$$Y_l^{mc}(\theta,\phi) \equiv \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\cos(m\phi). \tag{10}$$

The spherical harmonics obey

$$Y_l^{-l}(\theta,\phi) = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\sin^l\theta\, e^{-il\phi} \tag{11}$$

$$Y_l^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta) \tag{12}$$

$$Y_l^{-m}(\theta,\phi) = (-1)^m\bar{Y}_l^m(\theta,\phi), \tag{13}$$

where $P_l(x)$ is a Legendre polynomial.

Integrals of the spherical harmonics are given by

$$\int_0^{2\pi}\int_0^{\pi} Y_{l_1}^{m_1}(\theta,\phi)Y_{l_2}^{m_2}(\theta,\phi)Y_{l_3}^{m_3}(\theta,\phi)\sin\theta\,d\theta\,d\phi$$

$$= \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3\end{pmatrix}, \tag{14}$$

where $\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3\end{pmatrix}$ is a Wigner 3*j*-symbol (which is related to the Clebsch-Gordan coefficients).

Special cases include

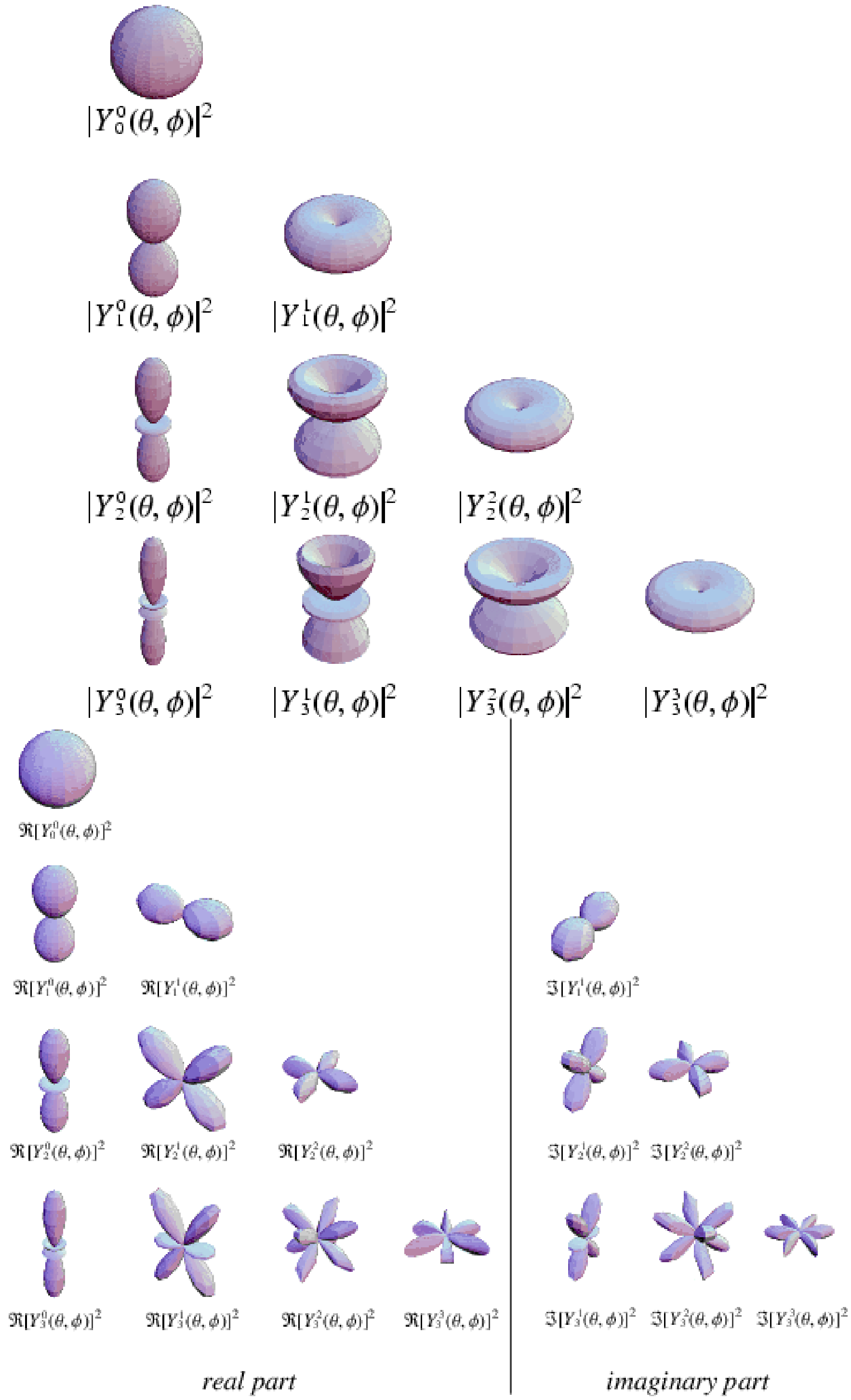
$$\int_0^{2\pi}\int_0^{\pi} Y_L^M(\theta,\phi)Y_0^0(\theta,\phi)\bar{Y}_L^M(\theta,\phi)\sin\theta\,d\theta\,d\phi = \frac{1}{\sqrt{4\pi}} \tag{15}$$

$$\int_0^{2\pi}\int_0^{\pi} Y_L^M(\theta,\phi)Y_1^0(\theta,\phi)\bar{Y}_{L+1}^M(\theta,\phi)\sin\theta\,d\theta\,d\phi = \sqrt{\frac{3}{4\pi}}\sqrt{\frac{(L+M+1)(L-M+1)}{(2L+1)(2L+3)}} \tag{16}$$

$$\int_0^{2\pi}\int_0^{\pi} Y_L^M(\theta,\phi)Y_1^1(\theta,\phi)\bar{Y}_{L+1}^{M+1}(\theta,\phi)\sin\theta\,d\theta\,d\phi = \sqrt{\frac{3}{8\pi}}\sqrt{\frac{(L+M+1)(L+M+2)}{(2L+1)(2L+3)}} \tag{17}$$

$$\int_0^{2\pi}\int_0^{\pi} Y_L^M(\theta,\phi)Y_1^1(\theta,\phi)\bar{Y}_{L-1}^{M+1}(\theta,\phi)\,\sin\theta\,d\theta\,d\phi = -\sqrt{\frac{3}{8\pi}}\sqrt{\frac{(L-M)(L-M-1)}{(2L-1)(2L+1)}} \quad (18)$$

(Arfken 1985, p. 700).



The above illustrations show $|Y_l^m(\theta,\phi)|^2$ (top), $\Re[Y_l^m(\theta,\phi)]^2$ (bottom left), and $\Im[Y_l^m(\theta,\phi)]^2$ (bottom right). The first few spherical harmonics are

$$Y_0^0(\theta,\phi) = \frac{1}{2}\frac{1}{\sqrt{\pi}}$$

$$
\begin{aligned}
Y_1^{-1}(\theta,\phi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{-i\phi} \\
Y_1^{0}(\theta,\phi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta \\
Y_1^{1}(\theta,\phi) &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{i\phi} \\
Y_2^{-2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{-2i\phi} \\
Y_2^{-1}(\theta,\phi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta\, e^{-i\phi} \\
Y_2^{0}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta-1) \\
Y_2^{1}(\theta,\phi) &= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta\, e^{i\phi} \\
Y_2^{2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi} \\
Y_3^{-3}(\theta,\phi) &= \frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta\, e^{-3i\phi} \\
Y_3^{-2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta\, e^{-2i\phi} \\
Y_3^{-1}(\theta,\phi) &= \frac{1}{8}\sqrt{\frac{21}{\pi}}\sin\theta(5\cos^2\theta-1)e^{-i\phi} \\
Y_3^{0}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{7}{\pi}}(5\cos^3\theta-3\cos\theta) \\
Y_3^{1}(\theta,\phi) &= -\frac{1}{8}\sqrt{\frac{21}{\pi}}\sin\theta(5\cos^2\theta-1)e^{i\phi} \\
Y_3^{2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta\, e^{2i\phi} \\
Y_3^{3}(\theta,\phi) &= -\frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta\, e^{3i\phi}.
\end{aligned}
$$

Written in terms of Cartesian coordinates,

$$e^{i\phi} = \frac{x+iy}{\sqrt{x^2+y^2}} \tag{19}$$

$$\theta = \sin^{-1}\left(\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}\right) \tag{20}$$

$$= \quad \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right), \tag{21}$$

so

$$Y_0^0(\theta,\phi) = \frac{1}{2}\frac{1}{\sqrt{\pi}} \tag{22}$$

$$Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{\sqrt{x^2+y^2+z^2}} \tag{23}$$

$$Y_1^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{\sqrt{x^2+y^2+z^2}} \tag{24}$$

$$Y_2^0(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(\frac{3z^2}{x^2+y^2+z^2}-1\right) \tag{25}$$

$$Y_2^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{z(x+iy)}{x^2+y^2+z^2} \tag{26}$$

$$Y_2^2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\frac{(x+iy)^2}{x^2+y^2+z^2}. \tag{27}$$

The zonal harmonics are defined to be those of the form

$$P_l^0(\cos\theta) = P_l(\cos\theta). \tag{28}$$

The tesseral harmonics are those of the form

$$\sin(m\phi)P_l^m(\cos\theta) \tag{29}$$

$$\cos(m\phi)P_l^m(\cos\theta) \tag{30}$$

for $l \neq m$. The sectorial harmonics are of the form

$$\sin(m\phi)P_m^m(\cos\theta) \tag{31}$$

$$\cos(m\phi)P_m^m(\cos\theta). \tag{32}$$

SEE ALSO: Condon-Shortley Phase, Correlation Coefficient, Laplace Series, Sectorial Harmonic, Solid Harmonic, Spherical Harmonic Addition Theorem, Spherical Harmonic Differential Equation, Spherical Harmonic Closure Relations, Spherical Vector Harmonic, Surface Harmonic, Tesseral Harmonic, Zonal Harmonic

RELATED WOLFRAM SITES: `http://functions.wolfram.com/Polynomials/SphericalHarmonicY/`, `http://functions.wolfram.com/HypergeometricFunctions/SphericalHarmonicYGeneral/`

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