Notes for Quantum Mechanics

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Lecture 26
Addition of Angular Momentum

Now I would like to talk about the addition of angular momentum. This will bring in one of the most important (and I think least understood) things in QM - that of the equivalence of different sets of eigenkets to describe a system. We have already seen this when we saw that we were free to choose the S_z or S_x basis, or the position or momentum basis. Now we will use what Liboff calls the uncoupled or coupled representation. (I don't like this differentiation)

This will lead to the Clebsch -Gordon Coefficients. I will not go through the process of showing you how these are calculated (I leave that for graduate school) but I want to show you what it means to add angular momentum - and via the Clebsch Gordon tables - show you practically how it is done and is used.

Remember I told you that we would have to be putting together the spacial part of the wave function - the thing we denoted in position representation as $\langle x'|\alpha\rangle$ and spin - which we denoted as $\langle +|\alpha\rangle$ or $\langle -|\alpha\rangle$. To be clear about it we often sometimes use a hybrid notation since matrix representation tends to be good for spin and we write it as $\begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix}$

We will get to this later.

Lets start with a simple example - that of adding two spins together. I will then go into a more general explanation of adding any two angular momenta together - in particular orbital and spin angular momentum.

Let us think about two electrons with spin $\frac{1}{2}$. We now need to denote them with *a* 1 or 2 for electron 1 and electron 2. (this will become a very important point later and lead us into the pauli exclusion principle). In any case we will denote the spin operators for the two electrons $\hat{S}_1$ and $\hat{S}_2$. We denote the total spin as $\hat{S}=\hat{S}_1+\hat{S}_2$.

Aside: Technically these things are actually two different hilbert spaces so using a funny notation we would formally write $\hat{S}=\hat{S}_1\otimes\hat{1}+\hat{1}\otimes\hat{S}_2$. The base kets for the two electron state can be written $|\pm_1,\pm_2\rangle = |\pm\rangle_1\otimes|\pm\rangle_2$ where the first space stands for electron 1 and the second for electron 2. The rotation operator around z is $\exp\left(-i\,\hat{J}_z\,\phi/\hbar\right)$ where now $\hat{J}=\hat{S}_1\otimes\hat{1}+\hat{1}\otimes\hat{S}_2$ and $\hat{S}_1$ and $\hat{S}_2$ only operate on their designated spaces.

This is important since when we make $\hat{J}=\hat{L}+\hat{S}$ we would have to write $\hat{J}=\hat{L}\otimes\hat{1}+\hat{1}\otimes\hat{S}$ where the first $\hat{1}$ has the two admissions of spin space and the second $\hat{1}$ has the infinite dimensions of orbital angular momentum space. These ideas also make it clear that the sum of two angular momentum operators make up another angular momentum operator . We can also write the rotation operator in this case as
$\exp\left(-i\,\hat{J}_z\,\phi/\hbar\right)=\exp\left(-i\,\hat{L}_z\,\phi/\hbar\right)\otimes\exp\left(-i\,\hat{S}_z\,\phi/\hbar\right)$

We have of course $[\hat{S}_{1x}, \hat{S}_{1y}]=i\hbar\,\hat{S}_{1z}$ and the same for y and z, and for electron 2. BUT we also now have

$[\hat{S}_{1x}, \hat{S}_{2y}]=0$. Operators operating on two different hilbert spaces commute. (suppose you were giving directions to two different cars. It doesn't matter if you tell one to turn left and then the other to turn right, or vice-versa)

After defining $\hat{S}=\hat{S}_1+\hat{S}_2$ we have $[\hat{S}_x, \hat{S}_y]=i\hbar\,\hat{S}_z$

proof: $[\hat{S}_x, \hat{S}_y]=[\hat{S}_{1x}, \hat{S}_{1y}]+[\hat{S}_{2x}, \hat{S}_{2y}]=i\hbar\,\hat{S}_{1z}+i\hbar\,\hat{S}_{2z}=i\hbar\,\hat{S}_z$

The eigenvalues are $\hat{S}^2 = \left(\hat{S}_1 + \hat{S}_2\right)^2 \to s(s+1)\,\hbar^2$ (this is the definition of s since $\hat{S}$ is a type of orbital angular momentum. It DOES NOT mean $s = s_1 + s_2$

$\hat{S}_z = \hat{S}_{1_z} + \hat{S}_{2_z} \to m\,\hbar \quad \hat{S}_{1z} \to m_1\,\hbar \quad \hat{S}_{2z} \to m_2\,\hbar \quad \hat{S}_1{}^2 \to s_1(s_1+1)\,\hbar^2 \quad \hat{S}_2{}^2 \to s_2(s_2+1)\,\hbar^2$

we know since we area working with electrons that $s_1 = s_2 = \frac{1}{2}$ and m_1 and m_2 are $\pm\frac{1}{2}$

Now comes a question. We have a 6 operators available to specify the state. Their eigenvalues are

s, s_1, s_2, s_z, s_{1_z}, s_{2z} How many do we need to specify? We might guess that since we are using a type of angular momentum S=J. then we can specify two. i.e. s and m and we already know $s_1 = s_2 = \frac{1}{2}$ since we are working with 2 electrons. Can we also specify m_{1z} and m_{2z}. It will turn out we cannot. We have two choices.

1) Specify the eigenvalues of $\hat{S}^2$ $\hat{S}_z$ $\hat{S}_1{}^2$ and $\hat{S}_2{}^2$ OR

2) Specify the eigenvalues of $\hat{S}_{1z}$ $\hat{S}_{2z}$ $\hat{S}_1{}^2$ and $\hat{S}_1{}^2$

Where we are thinking about adding angular momentum and spin i.e. $\hat{J}=\hat{L}+\hat{S}$ we can specify either :

1) the eigenvalues of $\hat{J}^2$ $\hat{J}_z$ $\hat{L}^2$ and $\hat{S}^2$ OR

2) the eigenvalues of $\hat{L}_z$, $\hat{S}_z$ $\hat{L}^2$ and $\hat{S}^2$

These are two different basis. We can choose either one.

You note that in both cases we specify $\hat{L}^2$ and $\hat{S}^2$. Typically we leave those off in lots of notation and this gets confusing

Lets get back to 2 spins. How can we make the relevant kets> First the kets which specify the values of $\hat{S}_{1z}$ and $\hat{S}_{2z}$ are $|m_1\rangle = |\pm\rangle$ and $|m_2\rangle = |\pm\rangle$. We can easily make the kets which are the two electron kets specifying these $|m_1\, m_2\rangle = |m_1\rangle\,|m_2\rangle$ e.g. $|{+}{+}\rangle = |{+}\rangle_1\,|{+}\rangle_2$. There must be 4 of these $|{+}{+}\rangle\ |{+}\,{-}\rangle\ |{-}\,{+}\rangle\ |{-}\,{-}\rangle$

But how can we make the one specifying $\hat{S}^2$ $\hat{S}_z$?

We can guess first that the maximum value of s can be 1, i.e. the sum of $\frac{1}{2} + \frac{1}{2}$. We might guess the same for m i.e. $m_{\max} = 1$ so this would give us

$|s{=}1, m = 1\rangle = |m_1 = +, m_2 = +\rangle$ (Got the notation??? I will write $|m_1 = +, m_2 = +\rangle$ as $|{+}{+}\rangle$ It is also written sometimes as $|m_1 = +\frac{1}{2}, m_2 = +\frac{1}{2}\rangle$) As before we can get at least some of the rest by using the lowering operator.

$\hat{S}_\pm = \hat{S}_{1\pm} + \hat{S}_{2\pm}$ This will give us practice. Remember $\hat{J}_\pm\,|j, m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}\,|j, m \pm 1\rangle$

$\hat{S}_-|s{=}1, m{=}1\rangle = (\hat{S}_{1-} + \hat{S}_{2-})|{+}{+}\rangle =$

For clarity I will rewrite this $\hat{S}_-|s{=}1, m{=}1\rangle = (\hat{S}_{1-} + \hat{S}_{2-})\,|s_1 = \frac{1}{2}\ m_1{=}+\ ,\ s_2 = \frac{1}{2}\ m_2{=}+\rangle$ So then we get

$\hat{S}_-|s=1, m=1\rangle = \hat{S}_{1-}\,|{+}{+}\rangle + \hat{S}_{2-}\,|{+}{+}\rangle =$

$\sqrt{(1+1)(1-1+1)}\,|s=1, m=0\rangle = \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\,|-+\rangle + \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\,|+-\rangle$

$|s=1, m=0\rangle = \frac{1}{\sqrt{2}}|-+\rangle + \frac{1}{\sqrt{2}}|+-\rangle$ Doing this again we get

$|s=1, m=-1\rangle = |--\rangle$ That's three, and to get the last one we look for one that is orthogonal to the other three

$|s=0, m=0\rangle = \frac{1}{\sqrt{2}}|+-\rangle - \frac{1}{\sqrt{2}}|-+\rangle$ Lets summarize these since we will need them

$$\left.\begin{array}{l} |\,s=1, m=1\rangle = |{+}{+}\rangle \\ |\,s=1, m=0\rangle = \frac{1}{\sqrt{2}}\,|{-}{+}\rangle + \frac{1}{\sqrt{2}}\,|{+}{-}\rangle \\ |\,s=1, m=-1\rangle = |{-}{-}\rangle \end{array}\right\} \quad s=1 \quad \text{triplet}$$

$$|\,s=0, m=0\rangle = \frac{1}{\sqrt{2}}|+-\rangle - \frac{1}{\sqrt{2}}|-+\rangle \qquad s=0 \quad \text{singlet}$$

General look at adding angular momentum $\hat{J} = \hat{J}_1 + \hat{J}_2$

We will start with two anymore momentum operators and their sum $\hat{J} = \hat{J}_1 + \hat{J}_2$

This is the angular momentum operator of the entire system we are worried about. Since it has to do with rotations its should also follow the commutation relations.

$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k \qquad [\hat{J}_{1i}, \hat{J}_{1j}] = i\hbar\epsilon_{ijk}\hat{J}_{1k} \qquad [\hat{J}_{2i}, \hat{J}_{2j}] = i\hbar\epsilon_{ijk}\hat{J}_{2k}$ BUT $[\hat{J}_{1i}, \hat{J}_{2j}] = 0$

also $[\hat{J}^2, \hat{J}_z] = 0 \quad [\hat{J}_1^{\,2}, \hat{J}_{1z}] = 0 \quad [\hat{J}_2^{\,2}, \hat{J}_{2z}] = 0 \quad \hat{J}_\pm = \hat{J}_{1\pm} + \hat{J}_{2\pm} \quad \hat{J}_\pm = \hat{J}_x + i\,\hat{J}_y$

$\hat{J}_\pm = \hat{J}_{1\pm} + \hat{J}_{2\pm} \quad \hat{J}_\pm = \hat{J}_x + i\,\hat{J}_y \quad \hat{J}_{1\pm} = \hat{J}_{1x} + i\,\hat{J}_{1y} \quad \hat{J}_{2\pm} = \hat{J}_{2x} + i\,\hat{J}_{2y}$

The rotation operator is $\exp\left(-i\,\hat{J}_z\,\phi/\hbar\right) = \exp\left(-i\,\hat{J}_{1z}\,\phi/\hbar\right) \otimes \exp\left(-i\,\hat{J}_{2z}\,\phi/\hbar\right)$. ϕ is the amount which is rotated and is the same for both 1 and 2.

We also have the relationship $[\hat{J}^2, \hat{J}_1^{\,2}]$

Proof: $[\hat{J}^2, \hat{J}_1^{\,2}] = [\hat{J}_1^{\,2} + \hat{J}_2^{\,2} + \hat{J}_1\hat{J}_2 + \hat{J}_2\hat{J}_1, \hat{J}_1^{\,2}] =$

$[\hat{J}_1^{\,2}, \hat{J}_1^{\,2}] + [\hat{J}_2^{\,2}, \hat{J}_1^{\,2}] + [\hat{J}_1\hat{J}_2, \hat{J}_1^{\,2}] + [\hat{J}_2\hat{J}_1, \hat{J}_1^{\,2}] =$

$\hat{J}_1[\hat{J}_2, \hat{J}_1^{\,2}] + [\hat{J}_1, \hat{J}_1^{\,2}]\hat{J}_2 + \hat{J}_2[\hat{J}_1, \hat{J}_1^{\,2}] + [\hat{J}_2, \hat{J}_1^{\,2}]\hat{J}_1 = 0$

Now as before we have two options to specify the states

1) Specify the eigenvalues of $\hat{J}_{1z}\ \hat{J}_{2z}\ \hat{J}_1^{\,2}\ \hat{J}_2^{\,2}$. These operators all commute with each other. We will denote these states as $|j_1\, j_2;\ m_1\, m_2\rangle$

$$\hat{J}_1^{\,2}\,|j_1\, j_2;\ m_1\, m_2\rangle = j_1(j_1+1)\,\hbar^2\,|j_1\, j_2;\ m_1\, m_2\rangle$$

$$\hat{J}_2^{\,2}\,|j_1\, j_2;\ m_1\, m_2\rangle = j_2(j_2+1)\,\hbar^2\,|j_1\, j_2;\ m_1\, m_2\rangle$$

$$\hat{J}_{1z}\,|j_1\, j_2;\ m_1\, m_2\rangle = m_1\,\hbar\,|j_1\, j_2;\ m_1\, m_2\rangle$$

$$\hat{J}_{2z}\,|j_1\, j_2;\ m_1\, m_2\rangle = m_2\,\hbar\,|j_1\, j_2;\ m_1\, m_2\rangle$$

2) Specify the eigenvalues of $\hat{J}^2\ \hat{J}_z\ \hat{J}_1^{\,2}\ \hat{J}_2^{\,2}$. These operators all commute with each other. We will denote these states as $|j_1\, j_2;\ j\, m\rangle$

$\hat{J}_1{}^2 \,|j_1\, j_2;\; j\, m\rangle = j_1(j_1+1)\,\hbar^2\,|j_1\, j_2;\; j\, m\rangle$

$\hat{J}_2{}^2 \,|j_1\, j_2;\; j\, m\rangle = j_2(j_2+1)\,\hbar^2\,|j_1\, j_2;\; j\, m\rangle$

$\hat{J}^2 \,|j_1\, j_2;\; j\, m\rangle = j(j+1)\,\hbar^2\,|j_1\, j_2;\; j\, m\rangle$

$\hat{J}_z \,|j_1\, j_2;\; j\, m\rangle = \mathrm{m}\hbar\,|j_1\, j_2;\; j\, m\rangle$

Now again since we need this often I will rewrite it in the notation that drops j_1 and j_2

$\hat{J} = \hat{J}_1 + \hat{J}_2$

1) $\hat{J}_{1z}\,|m_1\, m_2\rangle = m_1\,\hbar\,|m_1\, m_2\rangle$

$\hat{J}_{2z}\,|m_1\, m_2\rangle = m_2\,\hbar\,|m_1\, m_2\rangle$

2) $\hat{J}^2\,|\,j\, m\rangle = j(j+1)\,\hbar^2\,|\,j\, m\rangle$

$\hat{J}_z\,|\,j\, m\rangle = \mathrm{m}\hbar\,|\,j\, m\rangle$

Now both of these sets of eigenkets span the space. That means there must be a unitary transformation which allows us to go from one basis to another.

We know $\sum_{m_1}\,\sum_{m_2}\,|\,m_1\, m_2\rangle\langle m_1\, m_2|=1$

$|\,\mathrm{j\ m}\rangle=\sum_{m_1}\,\sum_{m_2}\,|\,m_1\, m_2\rangle\langle m_1\, m_2|\,\mathrm{j\ m}\rangle$ So to go back and forth we need to find $\langle m_1\, m_2|\,\mathrm{j\ m}\rangle$ These are the Clebsch-Gordon coefficients. Remember there is an implied j_1 and j_2

Now $m = m_1 + m_2$ proof: $\hat{J}_z\text{-}\hat{J}_{1z}\text{-}\hat{J}_{2z}=0$

look at the combination $\langle m_1\, m_2|\,\hat{J}_z\text{-}\hat{J}_{1z}\text{-}\hat{J}_{2z}|\mathrm{jm}\rangle=0$ and allow the operator to work backwards we get $m - m_1 - m_2=0$

Next we need to check the dimensionality of the space spanned by the two sets of eigenkets.

Its obvious that the $m_1\, m_2$ basis have a dimensionality $(2\, j_1 + 1)\,(2\, j_2 + 1)$ since m_i runs for $+\, j_i$ ..... $-\, j_i$

But if we do the same for the jm representation we will get too many. But if we think about the fact that J is a vector then we might guess that $|\, j_1 - j_2| \le \mathrm{j} \le |\, j_1 + j_2$ Lets check the dimensionality of such a space. We will assume $j_1 \ge j_2$

$N = \sum_{j=j_1-j_2}^{j=j_1+j_2} (2\, j + 1) = \frac{1}{2}\,\{[2\,(j_1 - j_2) + 1] + [2\,(j_1 + j_2) + 1]\}\,(2\, j_2 + 1) = (2\, j_1 + 1)(2\, j_2 + 1)$ so as long as we have the following - it works.

$|\, j_1 - j_2| \le \mathrm{j} \le |\, j_1 + j_2$

Some othogonality conditions : The $C - G$ coefficients form *a* unitary matrix –
and we take them to be real. This means that the inverse coefficients are the same *i.e.* we have

$\langle\ \mathrm{jm}\,|m_1\, m_2\rangle = \langle\ m_1\, m_2\,|\mathrm{jm}\rangle$ So we get

$$\sum_j \sum_m \langle\ m_1\, m_2\,|\mathrm{jm}\rangle\,\langle\ m_1{}'\, m_2{}'\,|\mathrm{jm}\rangle = \delta_{m_1\, m_1{}'}\ \ \delta_{m_2\, m_2{}'}$$

$$\sum_{m_1} \sum_{m_2} \langle\ m_1\, m_2\,|\mathrm{jm}\rangle\,\langle\ m_1\, m_2\,|\,j'\, m'\rangle = \delta_{\mathrm{jj}'}\,\delta_{\mathrm{mm}'}$$

$$\sum_{m_1} \sum_{m_2} |\ \langle m_1\, m_2\,|\mathrm{jm}\rangle\ |^2 = 1$$

Finally *a* last operator which we will need

$\hat{J} = \hat{L} + \hat{S}$ where S is spin

$\hat{L}\cdot\hat{S}\ = \frac{1}{2}\left(\hat{J}^2 - \hat{L}^2 - \hat{S}^2\right)$ The eigenvalues are

$$\frac{\hbar^2}{2}\left\{j(j+1) - l(l+1) - \frac{3}{4}\right\}\left\{\begin{array}{ll} l\,\frac{\hbar^2}{2} & \text{for } j = l + \frac{1}{2} \\ -(l+1)\,\frac{\hbar^2}{2} & \text{for } j = l = \frac{1}{2} \end{array}\right.$$