Notes for Quantum Mechanics

Richard Seto

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Lecture 24

Let's start with the fundamental commutator for angular momentum and get some of the properties of the eigenvalues. You will recognize the methods as similar to that which we used for spin and for the SHO

$[\hat{J}_i, \hat{J}_j]=i\hbar\epsilon_{ijk}\hat{J}_k$

Now define a total angular momentum $\hat{J}^2 \equiv \hat{J}_x^{\,2}+\hat{J}_y^{\,2}+\hat{J}_z^{\,2}$

We can show $[\hat{J}^2, \hat{J}_k]=0$

proof: for e.g. k=3

$[\hat{J}^2, \hat{J}_z]=[\hat{J}_x^{\,2}+\hat{J}_y^{\,2}+\hat{J}_z^{\,2}, \hat{J}_z]=\hat{J}_x[\hat{J}_x, \hat{J}_z]+[\hat{J}_x, \hat{J}_z]\hat{J}_x+\hat{J}_y[\hat{J}_y, \hat{J}_z]+[\hat{J}_y, \hat{J}_z]\hat{J}_y+\hat{J}_z[\hat{J}_z, \hat{J}_z]+[\hat{J}_z, \hat{J}_z]\hat{J}_z=$

$\hat{J}_x(-i\hbar\hat{J}_y)+(-i\hbar\hat{J}_y)\hat{J}_x+\hat{J}_y(i\hbar\hat{J}_x)+(i\hbar\hat{J}_x)\hat{J}_y+\hat{J}_z(0)+(0)\hat{J}_z=-i\hbar\hat{J}_x\hat{J}_y-i\hbar\hat{J}_y\hat{J}_x+i\hbar\hat{J}_y\hat{J}_x+i\hbar\hat{J}_x\hat{J}_y=0$

Now since $\hat{J}^2$ and $\hat{J}_z$ commute we will choose them to be simultaneously diagonalized. We could have chosen any of the $\hat{J}_k$ but we typically choose k=3. We will see that this requirement of simultaneous diagonalization will be necessary since some of the eigenvalues are degenerate. Let's call the eigenstate $|a,b\rangle$ where a is the eigenvalue of $\hat{J}^2$

i.e. $\hat{J}^2|a,b\rangle=a|a,b\rangle$ and b is the eigenvalue of $\hat{J}_z$ i.e. $\hat{J}_z|a,b\rangle=b|a,b\rangle$. (Later we will more conveniently set these to $a=\hbar^2 j(j+1)$ and $b=m\hbar$ where j and m are integers) Now both $\hat{J}^2$ and $\hat{J}_z$ will turn out to be observables so the eigenkets $|a,b\rangle$ are all orthogonal. (dont forget this)

As in the case of spin and also the SHO we will define

$\hat{J}_\pm \equiv \hat{J}_x+i\hat{J}_y$ then we have the following relations

$[\hat{J}_+, \hat{J}_-]=2\hbar\hat{J}_z$

$[\hat{J}_z, \hat{J}_\pm]=\pm\hbar\hat{J}_\pm$

$[\hat{J}^2, \hat{J}_\pm]=0$

Proof for the first of these.

$[\hat{J}_+, \hat{J}_-]=[\hat{J}_x+i\hat{J}_y, \hat{J}_x-i\hat{J}_y]=[\hat{J}_x, J_x]-i[\hat{J}_x, \hat{J}_y]+i[\hat{J}_y, \hat{J}_x]+[\hat{J}_y, \hat{J}_y]=-i(i\hbar\hat{J}_z)+i(-i\hbar\hat{J}_z)=2\hbar\hat{J}_z$

the second $[\hat{J}_z, \hat{J}_\pm]=[\hat{J}_z, \hat{J}_x \pm i\hat{J}_y]=[\hat{J}_z, \hat{J}_x]\pm i[\hat{J}_z, \hat{J}_y] = i\hbar\hat{J}_y \pm i(-i\hbar\hat{J}_x)=\pm\hbar(\hat{J}_x \pm i\hat{J}_y)=\pm\hbar\hat{J}_\pm$

and the third $[\hat{J}^2, \hat{J}_\pm]=[\hat{J}^2, \hat{J}_x \pm i\hat{J}_y]=[\hat{J}^2, \hat{J}_x]+ i[\hat{J}^2, \hat{J}_y]=0+0$ (from above)=0

Now with this lets see if we can figure out what $\hat{J}_\pm|a,b\rangle$ is.

Since $\hat{J}_z\hat{J}_\pm=\hat{J}_\pm\hat{J}_z \pm \hbar\hat{J}_\pm$ $\quad \hat{J}_z\hat{J}_\pm|a, b\rangle = (\hat{J}_\pm\hat{J}_z)|a, b\rangle \pm (\hbar\hat{J}_\pm)|a,b\rangle = \hat{J}_\pm b|a, b\rangle \pm \hbar\hat{J}_\pm|a,b\rangle = (b\pm\hbar)\hat{J}_\pm|a,b\rangle$

so we have $\hat{J}_z(\hat{J}_\pm|a, b\rangle)=(b\pm\hbar)(\hat{J}_\pm|a,b\rangle)$ so the eigenvalue for $\hat{J}_z$ of the ket $\hat{J}_\pm|a,b\rangle$ is $b\pm\hbar$ So $\hat{J}_\pm$ raises or lowers the value of b by $\hbar$.

What about $\hat{J}^2$? $\hat{J}^2(\hat{J}_\pm|a, b\rangle)=\hat{J}_\pm\hat{J}^2|a, b\rangle=\hat{J}_\pm a|a, b\rangle=a(\hat{J}_\pm|a, b\rangle)$ So the eigenvalue of $\hat{J}^2$ is not changed. Now there will be some sort of normalization constant c so we have that $\hat{J}_\pm|a,b\rangle=c|a,b\pm\hbar\rangle$. Now lets see if we can figure out what the values of a and b are. (We will get the normalization later)

First, lets prove that $b^2 < a$

$$\frac{1}{2}(\hat{J}_+\hat{J}_+^\dagger + \hat{J}_+^\dagger\hat{J}_+) = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) = \frac{1}{2}[(\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) + (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y)] =$$
$$\frac{1}{2}[\hat{J}_x^2 + \hat{J}_y^2 - i\hat{J}_x\hat{J}_y + i\hat{J}_y\hat{J}_x + \hat{J}_x^2 + \hat{J}_y^2 + i\hat{J}_x\hat{J}_y - i\hat{J}_y\hat{J}_x] = \hat{J}_x^2 + \hat{J}_y^2 = \hat{J}^2 - \hat{J}_z^2$$

now lets let $|\alpha\rangle=\hat{J}_+|a, b\rangle$ so $\langle\alpha|=\langle a,b|\hat{J}_+^\dagger$ so $\langle a,b|\hat{J}_+^\dagger\hat{J}_+|a, b\rangle=\langle\alpha|\alpha\rangle\geq 0 \implies$ any expectation value of $\hat{J}_+^\dagger\hat{J}_+ \geq 0$

now let $|\alpha\rangle=\hat{J}_+^\dagger|a, b\rangle$ so $\langle\alpha|=\langle a,b|\hat{J}_+$ so $\langle a,b|\hat{J}_+\hat{J}_+^\dagger|a, b\rangle=\langle\alpha|\alpha\rangle\geq 0 \implies$ any expectation value of $\hat{J}_+\hat{J}_+^\dagger \geq 0$

$\implies 0\leq\langle a,b|\ \frac{1}{2}(\hat{J}_+\hat{J}_+^\dagger + \hat{J}_+^\dagger\hat{J}_+)|a, b\rangle = \langle a, b|\hat{J}^2 - \hat{J}_z^2|a, b\rangle = \langle a, b|a - b^2|a, b\rangle =$

$(a - b^2)\langle a, b|a, b\rangle = a - b^2\geq 0 \implies b^2\leq a$

This means that there must be some $b_{\max}$ such that $\hat{J}_+|a, b_{\max}\rangle=0$ since $\hat{J}_+$ will just add and $\hbar$ to b, otherwise at some point b^2 will end up greater than a.

Applying $\hat{J}_-$ we get $\hat{J}_-\hat{J}_+|a, b_{\max}\rangle=0$ also

$\hat{J}_-\hat{J}_+ =(\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y)=\hat{J}_x^2 + \hat{J}_y^2 + i\hat{J}_x\hat{J}_y - i\hat{J}_y\hat{J}_x =\hat{J}^2 - \hat{J}_z^2+i[\hat{J}_x, \hat{J}_y]=$

$\hat{J}^2 - \hat{J}_z^2+i(i\hbar\hat{J}_z)=\hat{J}^2 - \hat{J}_z^2-\hbar\hat{J}_z$

So $0=\hat{J}_-\hat{J}_+|a, b_{\max}\rangle=(\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)|a, b_{\max}\rangle=(a - b_{\max}^2 - \hbar b_{\max})|a, b_{\max}\rangle$ So

$a = b_{\max}(b_{\max} + \hbar)$

Similarly there must be some $b_{\min}$ such that $\hat{J}_-|a, b_{\min}\rangle=0$

Applying $\hat{J}_+$ we get $\hat{J}_+\hat{J}_-|a, b_{\min}\rangle=0$ also

$\hat{J}_+\hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) =\hat{J}_x^2 + \hat{J}_y^2 - i\hat{J}_x\hat{J}_y + i\hat{J}_y\hat{J}_x =\hat{J}^2 - \hat{J}_z^2-i[\hat{J}_x, \hat{J}_y]=$

$\hat{J}^2 - \hat{J}_z^2-i(i\hbar\hat{J}_z)=\hat{J}^2 - \hat{J}_z^2+\hbar\hat{J}_z$

So $0=\hat{J}_+\hat{J}_-|a, b_{\min}\rangle=(\hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z)|a, b_{\min}\rangle=(a - b_{\min}^2 + \hbar b_{\min})|a, b_{\min}\rangle$ So

$a = b_{\min}(b_{\min} - \hbar)$

So putting the two together we get $a = b_{\max}(b_{\max} + \hbar) = b_{\min}(b_{\min} - \hbar)$ The solution to this is that $b_{\min} = -b_{\max}$

Now you must be able to get from $b_{\min}$ to $b_{\max}$ using the raising operator and incementing b by h so $b_{\max} = b_{\min} + N\hbar$

where N is an integer. Using $b_{\min} = -b_{\max}$ we get $-b_{\max} = b_{\max} - N\hbar$ so we get that

$b_{\max} = \frac{N\hbar}{2}$ now define $j \equiv \frac{b_{\max}}{\hbar} = \frac{N}{2}$ and using $a = b_{\max}(b_{\max} + \hbar) = j\hbar(j\hbar + \hbar) = \hbar^2\, j(j+1)$

So we get that j is integer or half integer

We are now forced into a situation in which the angular momentum eigenvalues are $\hbar^2\, j(j+1)$ where j is either integer or half-integer. The half integer things will be spin. The integer values will correspond to orbital angular momentum

Now this will all work if $b = m\hbar$ m=-j,-j+1,-j+2,...+j that is there are 2j+1 states and we will call $|a,b\rangle \longrightarrow |j,m\rangle$

$\hat{J}^2\, |j, m\rangle = j(j+1)\,\hbar^2 |j,m\rangle$ j is integer or half integer

$\hat{J}_z\, |j, m\rangle = m\hbar |j,m\rangle$ m=-j,-j+1,-j+2,...+j

Now for good measure lets find the normalization for $\hat{J}_+\, |j, m\rangle$ assuming $|j, m\rangle$ is correctly normalized

let $\hat{J}_\pm\, |j, m\rangle = c\, |j, m \pm 1\rangle$ where c is the normalization constant

$$1 = \langle j, m|\, \hat{J}_\pm^\dagger\, \hat{J}_\pm\, |j, m\rangle = c^2 \langle j, m \pm 1\, |j, m \pm 1\rangle = c^2 = \langle j, m|\, \hat{J}_\mp\, \hat{J}_\pm\, |j, m\rangle =$$

$$\langle j, m|\, \hat{J}^2 - \hat{J}_z^{\,2} \pm \hbar\, \hat{J}_z\, |j, m\rangle = \hbar^2 [j(j+1) \pm m(m+1)]\, \langle j, m|\, j, m\rangle \text{ so}$$

$$\hat{J}_\pm\, |j, m\rangle = \hbar \sqrt{(j+1) \pm m(m+1)}\; |j, m \pm 1\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)}\; |j, m \pm 1\rangle$$

So we can summarize our logic.

1) We looked at rotations and define $\hat{J}$ as the "generator" or rotations i.e. $D(\epsilon) = 1 - i\epsilon\hat{J}$

2) this gave us the commutations relationships for J

3) This leads to the quantization of J into integer or half integer. i.e. we get spin!

Now we are going to use matrix notation to denote the eigenstates of angular momentum

$$\langle j', m'|\, \hat{J}^2\, |j, m\rangle = j(j+1)\,\hbar^2\, \delta_{j'j}\, \delta_{m'm}$$

$$\langle j', m'|\, \hat{J}_z\, |j, m\rangle = m\hbar\, \delta_{j'j}\, \delta_{m'm}$$

$$\langle j', m'|\, \hat{J}_\pm\, |j, m\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)}\; \delta_{j'j}\, \delta_{m'm\pm 1}$$

Now we can figure out the matrix elements of the rotation operator

$$\hat{D}^{j'j}{}_{m'm}(\hat{n}, \phi) = \langle j', m'|\, \exp\left(-\frac{i}{\hbar}\, \hat{J} \cdot \hat{n}\, \phi\right) |j, m\rangle$$

Now since the $\hat{J}_k$ all commute with $\hat{J}^2$ then $\hat{D}(\hat{n}, \phi) = \exp\left(-\frac{i}{\hbar}\, \hat{J} \cdot \hat{n}\, \phi\right)$ commutes with $\hat{J}^2$ so $\hat{J}^2\, \hat{D}(\hat{n}, \phi)|j,m\rangle = \hat{D}(\hat{n}, \phi)\, \hat{J}^2 |j,m\rangle = \hat{D}(\hat{n}, \phi)\, j(j+1)\,\hbar^2 |j,m\rangle = j(j+1)\,\hbar^2 \hat{D}(\hat{n}, \phi)|j,m\rangle$ so $\hat{D}(\hat{n}, \phi)|j,m\rangle$ is an eigenket of $\hat{J}^2$ with the same eigenvalue as j as $|j,m\rangle$ $\Longrightarrow$ in simple terms, rotations do not change the total angular momentum. They will however change the z-component of angular momentum, i.e. the eigenvalues of $\hat{J}_z$ This means that rotations cannot mix up different values of j. So $\langle j', m'|\, \exp\left(-\frac{i}{\hbar}\, \hat{J} \cdot \hat{n}\, \phi\right) |j, m\rangle$ will be $\sim \delta_{j'j}$

Now we will order the elements of our matrix by starting from the lowest value of j possible. Since j has to be integer or half integer then the lowest $j = \frac{1}{2}$ and go through the m values $m = -\frac{1}{2}, \frac{1}{2}$ the next is j=1 and m= -1,0,1, then

$j = \frac{3}{2}$, $m = \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}$ etc This of course goes on forever since j is some integer or half integer, with no limit. In practice we will look at systems with a particular value of j, so the matrix notation can be useful

The matrix below has an x in the non-zero elements

$$\begin{pmatrix} x & x & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ x & x & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & x & x & x & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & x & x & x & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & x & x & x & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & x & x & x & x & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & x & x & x & x & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & x & x & x & x & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & x & x & x & x & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square & \square & x & x & x & x & x \\ \square & \square & \square & \square & \square & \square & \square & \square & \square & x & x & x & x & x \\ \square & \square & \square & \square & \square & \square & \square & \square & \square & x & x & x & x & x \\ \square & \square & \square & \square & \square & \square & \square & \square & \square & x & x & x & x & x \\ \square & \square & \square & \square & \square & \square & \square & \square & \square & x & x & x & x & x \end{pmatrix}$$