Notes for Quantum Mechanics

Richard Seto

Last revised: Sep. 24, 2004

Lecture 11

```
Date[]
```

```
{2004, 11, 9, 9, 35, 32.2311712}
```

Lecture 11

Lets now try doing a calculation with what we have learned. First lets just put an electron in a magnetic field and then figure out what the potential energy from the spin-magnetic field interaction is.

First we know that U=-$\vec{\mu}\cdot\vec{B}$. Now we can relate the magnetic moment to the spin. This is a physics 40 problem. Try it. One thing you have to worry about is the units so be careful. In any case the answer will be $\vec{\mu} = \frac{-e}{\mathrm{mc}}\vec{S}$.

So what is $\vec{S}$? Well, in our new way of doing things, its an operator. So U now becomes an operator which we often call the Hamiltonian $\hat{H}$. So lets rewrite it $\hat{U}$=$\hat{H}$=$\frac{e}{\mathrm{mc}}\,\vec{\hat{S}}\cdot\vec{B}$ where we have now written the spin operator as

$\vec{\hat{S}} = (\hat{S}_x,\ \hat{S}_y,\ \hat{S}_z) = \frac{\hbar}{2}\,(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ Lets think about this. First this is now an OPERATOR. Secondly is a vector (in regular x,y,z space) So its a vector operator. Finally, it assumes that we are living also in a spin space, i.e. $|+\rangle$ and $|-\rangle$. DON'T GET spin space(with 2 directions -or dimensions) and x,y,z space (with 3 directions -or dimensions) mixed up.

so $\hat{U}$=$\hat{H}$=$\frac{e\hbar}{2\,\mathrm{mc}}$ $(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)\cdot\vec{B} = \frac{-e\hbar}{2\,\mathrm{mc}}$ $(\hat{\sigma}_x\, B_x + \hat{\sigma}_y\, B_y + \hat{\sigma}_z\, B_z\,)$

Note that we are now using the matrix form of the operator! Try writing it out in the form using bra's and kets.

OK back to $\hat{H}$. Now the components of the magnetic field B dont have spin degrees of freedom so they can just be pulled through the pauli spin matrices and we can just write

$\hat{H}=\frac{e\hbar}{2\,\mathrm{mc}}$ $(B_x\, \hat{\sigma}_x + B_y \hat{\sigma}_y + B_z \hat{\sigma}_z\,)$

So now lets find the expectation value of $\hat{H}$ for a spin up particle i.e. $|+\rangle$, so we want to find $\langle + \mid \hat{H} \mid + \rangle$. Let's work in matrix notation so we need to find

x-part $\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}=0$

y-part $\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ i \end{pmatrix}=0$

z-part $\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=1$

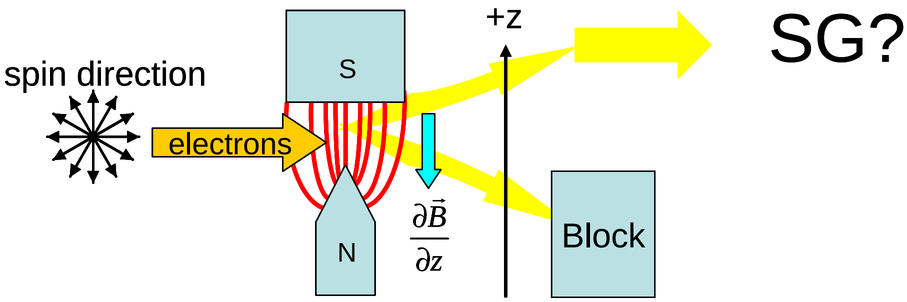
so finally $\langle + \mid \hat{H} \mid + \rangle = \frac{e\hbar\ B_z}{2\,\mathrm{mc}}$

The combination $\frac{e\hbar}{2\,\mathrm{mc}} = 928.476\,362 \times 10^{-26}\ \mathrm{J\,T^{-1}} = 5.796 \times 10^{-5}\ \mathrm{eV/T}$

So if it is in a $B_z = 1$ T field, the difference between spin up and spin down will be 11.6×10^{-5} eV

A real experiment (with its associated problems)

OK. You set up a Stern-Gerlach experiment in your lab. Remember that the force $F_z = \frac{d}{dz}(\vec{\mu}\cdot\vec{B})$. In practice you can't make a magnetic field that has ONLY as B_z component.There will always be a little B_x and B_y (for that matter you can't make one with only a derivative in the z direction - but lets ignore that for now). Lets assume that we have a small B_x component. So $\frac{d\vec{B}}{dz}$=(δb,0,b) where δ is the small percentage of leakage and b is $\frac{dB_z}{dz}$. We then block the stuff where the force is downward. The question is - what is the state which passes on to the next part of the experiment. It is no longer SGz+.



Now before I do this harder problem, lets review what we might do in the simplest case where $\frac{d\vec{B}}{dz}$=(0,0,b) . We will take $\hat{F}_z$ to be an operator which makes a measurement. We then want to find the force on the beam and see which ones are pushed up and which ones are pushed down. This means that we want to find the the eigenfunctions of $\hat{F}_z$ where the positive eigenvalues will tell me the force upward and the negative eigenvalues will tell me the force downward. Lets do it.

We then write $\hat{F}_z = b\,\hat{\mu}_z$ where this thing is now an operator. So then we want to solve

$\hat{F}_z|\alpha\rangle = b\,\frac{e}{\mathrm{mc}}\,\hat{S}_z|\alpha\rangle = \mathrm{f}|\alpha\rangle$ where f are the eigenvalues. Lets use matrix notation (also do this your self using bra's and kets)

$b\,\frac{e\hbar}{2\,\mathrm{mc}}\,\hat{\sigma}_z\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \mathrm{f}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ now we already know the eigenkets of $\hat{\sigma}_z$ which are $\begin{pmatrix}1\\0\end{pmatrix}$ with an eigenvalue of +1 and $\begin{pmatrix}0\\1\end{pmatrix}$ with an eigenvalue of -1 but lets solve it so we can learn

$b\,\frac{e\hbar}{2\,\mathrm{mc}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \mathrm{f}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ We can rewrite this using the identity matrix $\hat{1} \doteq \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ as $-b\,\frac{e\hbar}{\mathrm{mc}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \mathrm{f}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$

and define $f' = \frac{2\,\mathrm{mc}}{be\hbar}$ so then $\begin{pmatrix}1-f' & 0\\ 0 & -1-f'\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=0$ so lets first solve for the eigenvalues

$\begin{vmatrix}1-f' & 0\\ 0 & -1-f'\end{vmatrix}=0$ so (1-f')(-f'-1)=0 so f=±1 or f=± $b\,\frac{e\hbar}{2\,\mathrm{mc}}$ which is what we expected. Now lets sove for $\alpha_1$and α_2

$\begin{pmatrix}\alpha_1(1-f')\\ -\alpha_2(1+f')\end{pmatrix}=0$ So for f'=+1 $\alpha_2 = 0$ but we also have the normalization condition $\sqrt{\alpha_1{}^2+\alpha_2{}^2}=1$ giving α_1=1 (or -1 which we will ignore for now) so the eigenfunction corresponding to the eigenvalue f'=+1 (f=+$\frac{\text{be}\hbar}{2\,\text{mc}}$) is $\begin{pmatrix}1\\0\end{pmatrix}$. If we do the same for f'=-1 (f=-$\frac{\text{be}\hbar}{2\,\text{mc}}$) we will get $\begin{pmatrix}0\\1\end{pmatrix}$ as we had thought.

Now lets do the harder problem $\frac{d\vec{B}}{\text{dz}} = (\delta\text{b}, 0, b)$ so $\hat{F}_z = \vec{\mu}\cdot\frac{d\vec{B}}{\text{dz}} = \delta\text{b}\,\hat{\mu}_x + b\,\hat{\mu}_z$ We will have to solve

$b\,\frac{e\hbar}{2\,\text{mc}}[\,\delta\,\hat{\sigma}_x + \hat{\sigma}_z]\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}=\text{f}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}$ $\quad$ $[\delta\begin{pmatrix}0&1\\1&0\end{pmatrix}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}]\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}=\text{f}'\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}$

$\begin{pmatrix}1-f' & \delta\\ \delta & -1-f'\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}=0$ $\quad$ so now lets find the eigenvalues

$(1-f')(-f'-1)-\delta^2=0 \qquad (f'-1)(f'+1)-\delta^2 = 0$

$f'^2-1-\delta^2=0 \quad f'_\pm = \pm\sqrt{1+\delta^2}$ (f=$\pm\frac{\text{be}\hbar}{2\,\text{mc}}\sqrt{1+\delta^2}$) So you see the force is just a little bit bigger because of the extra field in the x direction. Now we find the eigenvectors:

$\begin{pmatrix}\alpha_1(1-f')+\delta\alpha_2\\ \delta\alpha_1-\alpha_2(1+f')\end{pmatrix}=0$ now this means $\alpha_1 = \frac{\alpha_2(1+f)}{\delta}$ (using the bottom eqn). Lets set α_2=1 and then normalize later

Expanding $\sqrt{1+\delta^2} \sim 1+\frac{\delta^2}{2}+...$

So we get unnomalized

$\begin{pmatrix}\frac{(1+f')}{\delta}\\1\end{pmatrix}$ So for f'_- we get $\begin{pmatrix}\frac{\left(1-\sqrt{1+\delta^2}\right)}{\delta}\\1\end{pmatrix}\sim\begin{pmatrix}\frac{1-\left(1+\frac{\delta^2}{2}\right)}{\delta}\\1\end{pmatrix}=\begin{pmatrix}\frac{-\frac{\delta^2}{2}}{\delta}\\1\end{pmatrix}=\begin{pmatrix}-\frac{\delta}{2}\\1\end{pmatrix}$ This is already normalized to first order

Expanding $\sqrt{1+\delta^2} \sim 1+\frac{\delta^2}{2}+...$

for f'_+ $\begin{pmatrix}\frac{\left(1+\sqrt{1+\delta^2}\right)}{\delta}\\1\end{pmatrix}\sim\begin{pmatrix}\frac{1+\left(1+\frac{\delta^2}{2}\right)}{\delta}\\1\end{pmatrix}=\begin{pmatrix}\frac{2+\frac{\delta^2}{2}}{\delta}\\1\end{pmatrix}=\begin{pmatrix}\frac{2}{\delta}+\frac{\delta}{2}\\1\end{pmatrix}$ Now we have to renormalize this i.e. $\alpha_1{}^2+\alpha_2{}^2=1$

so the normalization constant= $\frac{1}{\sqrt{\alpha_1{}^2+\alpha_2{}^2}}=\frac{1}{\sqrt{\left(\frac{2}{\delta}+\frac{\delta}{2}\right)^2+1}} = \frac{1}{\sqrt{\frac{4}{\delta^2}+2+\frac{\delta^2}{4}+1}} = \frac{\delta}{\sqrt{4+3\,\delta^2+\frac{\delta^4}{4}}}\sim\frac{\delta}{2}$ so for f_+

$\frac{\delta}{2}\begin{pmatrix}\frac{2}{\delta}+\frac{\delta}{2}\\1\end{pmatrix}=\begin{pmatrix}1\\\frac{\delta}{2}\end{pmatrix}$

from mathematica

So the eigenket for $f'_+=+\sqrt{1+\delta^2}$ $\left(f = +\frac{\text{be}\hbar}{2\,\text{mc}}\sqrt{1+\delta^2}\right)$ is $\begin{pmatrix}1\\\frac{\delta}{2}\end{pmatrix}$ and for $f'_-=-\sqrt{1+\delta^2}$ $\left(f = -\frac{\text{be}\hbar}{2\,\text{mc}}\sqrt{1+\delta^2}\right)$ is $\begin{pmatrix}-\frac{\delta}{2}\\1\end{pmatrix}$

So now lets check it

$\begin{pmatrix}1&\delta\\\delta&-1\end{pmatrix}\begin{pmatrix}1\\\frac{\delta}{2}\end{pmatrix}=\begin{pmatrix}1+\frac{\delta^2}{2}\\\delta-\frac{\delta}{2}\end{pmatrix}=\begin{pmatrix}1+\frac{\delta^2}{2}\\\frac{\delta}{2}\end{pmatrix}\sim(1+\frac{\delta^2}{2})\begin{pmatrix}1\\\frac{\delta}{2}\end{pmatrix}$ so this is the one with a positive eigenvalue

$\begin{pmatrix}1&\delta\\\delta&-1\end{pmatrix}\begin{pmatrix}-\frac{\delta}{2}\\1\end{pmatrix}=\begin{pmatrix}-\frac{\delta}{2}+\delta\\-1-\frac{\delta^2}{2}\end{pmatrix}=\begin{pmatrix}\frac{\delta}{2}\\-\left(1+\frac{\delta^2}{2}\right)\end{pmatrix}\sim-(1+\frac{\delta^2}{2})\begin{pmatrix}-\frac{\delta}{2}\\1\end{pmatrix}$ and the one with the negative eigenvalue

For those using mathematica

```
a = {{1, d}, {d, -1}}
```

$\{\{1, d\}, \{d, -1\}\}$

```
Eigenvalues[a]
```

$\left\{-\sqrt{1+d^2}, \sqrt{1+d^2}\right\}$

```
Eigenvectors[a]
```

$\left\{\left\{-\frac{-1+\sqrt{1+d^2}}{d}, 1\right\}, \left\{-\frac{-1-\sqrt{1+d^2}}{d}, 1\right\}\right\}$

$$\mathbf{ap} = \left\{-\frac{-1+\sqrt{1+d^2}}{d}, 1\right\}$$

$\left\{-\frac{-1+\sqrt{1+d^2}}{d}, 1\right\}$

```
Series[ap, {d, 0, 1}]
```

$\left\{-\frac{d}{2} + O[d]^2, 1\right\}$

$$\mathbf{am} = \left\{-\frac{-1-\sqrt{1+d^2}}{d}, 1\right\}$$

$\left\{-\frac{-1+\sqrt{1+d^2}}{d}, 1\right\}$

```
Series[am, {d, 0, 1}]
```

$$\left\{\frac{2}{d}+\frac{d}{2}+O[d]^2,\ 1\right\}$$

```
amn = am / Norm[am]
```

$$\left\{-\frac{-1-\sqrt{1+d^2}}{d\sqrt{1+\mathrm{Abs}\left[\frac{-1-\sqrt{1+d^2}}{d}\right]^2}},\ \frac{1}{\sqrt{1+\mathrm{Abs}\left[\frac{-1-\sqrt{1+d^2}}{d}\right]^2}}\right\}$$

$$\mathbf{amn2} = \left\{-\frac{-1-\sqrt{1+d^2}}{d\sqrt{1+\left(\frac{-1-\sqrt{1+d^2}}{d}\right)^2}},\ \frac{1}{\sqrt{1+\left(\frac{-1-\sqrt{1+d^2}}{d}\right)^2}}\right\}$$

$$\left\{-\frac{-1-\sqrt{1+d^2}}{d\sqrt{1+\frac{\left(-1-\sqrt{1+d^2}\right)^2}{d^2}}},\ \frac{1}{\sqrt{1+\frac{\left(-1-\sqrt{1+d^2}\right)^2}{d^2}}}\right\}$$

```
Series[amn2, {d, 0, 1}]
```

$$\left\{1+O[d]^2,\ \frac{d}{2}+O[d]^2\right\}$$