Physics 156A Quantum Mechanics - Final

December 10, 2004

Solutions

Date[]

{2004, 12, 14, 14, 4, 22.1224576}

First – My apologies – there was *a* typo

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Total - 100 pts
pr 1 - 25 pts
pr 2 - 25 pts
pr 3 - 25 pts
pr 4 - 25 pts
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1) You have a wave function for a particle in a box with sides at 0 and a. The wave function is in the n=3 energy eigenstate. That is $\langle x|n=3\rangle = \psi_3(x) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{3\pi x}{a}\right)$

a) Find the corresponding momentum representation of this wavefunction, i.e. $\langle p|n=3\rangle = \varphi_3(p)$

Note : $e^{i3\pi} = \cos 3\pi + i\sin 3\pi = -1 \qquad e^{-i3\pi} = \cos(-3\pi) + i\sin(-3\pi) = -1$

$$\varphi(k) = \frac{1}{\sqrt{2\pi}} \int_0^a e^{ikx'} \psi(x')\, dx' = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{a}} \int_0^a dx'\, e^{ikx'} \sin\left(\frac{3\pi x'}{a}\right) =$$

$$= \frac{1}{\sqrt{\pi a}} \frac{1}{2i} \int_0^a dx'\, e^{ikx'} \left[e^{\frac{i3\pi x'}{a}} - e^{-\frac{i3\pi x'}{a}}\right] =$$

$$\frac{1}{\sqrt{\pi a}} \frac{1}{2i} \int_0^a dx' \left[e^{i(k+3\frac{\pi}{a})x'} - e^{i(k-3\frac{\pi}{a})x'}\right] =$$

$$\frac{1}{\sqrt{\pi a}} \frac{1}{2i} \left[\frac{e^{i(k+3\frac{\pi}{a})x'}}{i(k+3\frac{\pi}{a})} - \frac{e^{i(k-3\frac{\pi}{a})x'}}{i(k-3\frac{\pi}{a})}\right] =$$

$$\frac{-1}{2\sqrt{\pi a}} \left[\frac{e^{i(k+3\frac{\pi}{a})a}}{(k+3\frac{\pi}{a})} - \frac{e^{i(k-3\frac{\pi}{a})a}}{(k-3\frac{\pi}{a})} - \frac{1}{(k+3\frac{\pi}{a})} + \frac{1}{(k-3\frac{\pi}{a})}\right]$$

$$= \frac{-1}{2\sqrt{\pi a}} \left[\frac{e^{ika}\, e^{i3\pi} - 1}{(k+3\frac{\pi}{a})} - \frac{e^{ika}\, e^{-i3\pi} - 1}{(k-3\frac{\pi}{a})}\right] = \frac{1}{2\sqrt{\pi a}} \left[\frac{e^{ika} + 1}{(k+3\frac{\pi}{a})} - \frac{e^{ika} + 1}{(k-3\frac{\pi}{a})}\right] =$$

$$= \frac{(e^{ika} + 1)}{2\sqrt{\pi a}} \left[\frac{1}{(k+3\frac{\pi}{a})} - \frac{1}{(k-3\frac{\pi}{a})}\right] =$$

$$\frac{(e^{ika} + 1)}{2} \sqrt{\frac{a}{\pi}} \left[\frac{1}{(ka+3\pi)} - \frac{1}{(ka-3\pi)}\right] =$$

$$= \frac{(e^{ika} + 1)}{2} \sqrt{\frac{a}{\pi}} \left[\frac{(ka-3\pi) - (ka+3\pi)}{k^2 a^2 - 9\pi^2}\right] =$$

$$\frac{(e^{ika} + 1)}{2} \sqrt{\frac{a}{\pi}} \left[\frac{-6\pi}{k^2 a^2 - 9\pi^2}\right] =$$

$$= \frac{-3\,(1 + e^{ika})\sqrt{\pi a}}{k^2 a^2 - 9\pi^2}$$

b) Find Δx and Δp. Do they agree with the heisenberg uncertainty principle?

$\langle x\rangle = \frac{a}{2}$ by symmetry or

$$\langle x\rangle = \frac{2}{a}\int_0^a dx\, x \sin^2\left(\frac{3\pi x}{a}\right) = \frac{2}{a}\left(\frac{x^2}{4} - \frac{x\sin\left(\frac{6\pi x}{a}\right)}{\frac{12\pi}{a}} - \frac{\cos\left(\frac{6\pi x}{a}\right)}{\frac{72\pi^2}{a^2}}\right) =$$

$$\frac{2}{a}\left(\frac{a^2}{4} - \frac{x\sin(6\pi)}{\frac{12\pi}{a}} - \frac{\cos(6\pi)}{\frac{72\pi^2}{a^2}} + \frac{1}{\frac{72\pi^2}{a^2}}\right) = \frac{2}{a}\,\frac{a^2}{4} = \frac{a}{2}$$

$\langle x^2\rangle =$

$$\frac{2}{a}\int_0^a dx\, x^2 \sin^2\left(\frac{3\pi x}{a}\right) = \frac{2}{a}\left(\frac{x^3}{6} - \left(\frac{x^2}{\frac{12\pi}{a}} - \frac{1}{\frac{72\pi^2}{a^2}}\right)\sin\left(\frac{6\pi x}{a}\right) - \frac{x\cos\left(\frac{6\pi x}{a}\right)}{\frac{36\pi^2}{a^2}}\right) =$$

$$= \frac{2}{a}\left(\frac{a^3}{6} - \frac{a}{\frac{36\pi^2}{a^2}}\right) = \frac{a^2}{3} - \frac{a^2}{18\pi^2}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{3} - \frac{a^2}{18\pi^2} - \frac{a^2}{4}} = a\sqrt{\frac{1}{12} - \frac{1}{18\pi^2}}$$

$\langle p\rangle =$

$$\frac{2}{a}\,\frac{\hbar}{i}\int_0^a dx\, \sin\left(\frac{3\pi x}{a}\right)\frac{d}{dx}\sin\left(\frac{3\pi x}{a}\right) = \frac{2}{a}\,\frac{\hbar}{i}\int_0^a dx\, \sin\left(\frac{3\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)\frac{3\pi}{a} = 0$$

$$\langle p^2\rangle = -\hbar^2\,\frac{2}{a}\int_0^a dx\, \sin\left(\frac{3\pi x}{a}\right)\frac{d^2}{dx^2}\sin\left(\frac{3\pi x}{a}\right) =$$

$$\hbar^2\,\frac{2}{a}\left(\frac{3\pi}{a}\right)^2\int_0^a dx\, \sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right) =$$

$$\hbar^2\,\frac{2}{a}\left(\frac{3\pi}{a}\right)^2\left(\frac{x}{2} - \frac{1}{4\left(\frac{3\pi}{a}\right)}\sin\left(\frac{6\pi x}{a}\right)\right) = \hbar^2\,\frac{2}{a}\left(\frac{3\pi}{a}\right)^2\left(\frac{a}{2}\right) = \hbar^2\left(\frac{3\pi}{a}\right)^2$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\hbar^2\left(\frac{3\pi}{a}\right)^2} = \frac{3\pi\hbar}{a}$$

$$\Delta x\Delta p = a\sqrt{\frac{1}{12} - \frac{1}{18\pi^2}}\,\frac{3\pi\hbar}{a} = \sqrt{\frac{3\pi^2}{4} - \frac{1}{2}}\,\hbar = \frac{\hbar}{2}\sqrt{3\pi^2 - 2}$$

which is greater than $\frac{\hbar}{2}$ so it satisfies heisenberg

c) In the basis of energy eigenkets, give the matrix representations of $\hat{H}$, $\hat{p}$, and $\hat{x}$. (note that it is infinite in dimension - just give the first 3x3 part and indicate how the matrix continues)

c) **We know that H is diagonal with eigenvalues $E_n = \frac{n^2 \hbar^2 \pi^2}{2\,\mathrm{ma}^2}$ so**

$$H = \begin{pmatrix} 1 & 0 & 0 & \square \\ 0 & 4 & 0 & \square \\ 0 & 0 & 9 & 0 \\ \square & \square & 0 & 4^4 \end{pmatrix} \frac{\hbar^2 \pi^2}{2\,\mathrm{ma}^2}$$

for p the matrix elements are $\langle n \mid \hat{p} \mid m\rangle$ where $\langle x \mid n\rangle = \sqrt{\frac{2}{a}}\,\sin\left(\frac{n\pi x}{a}\right)$

$\langle n \mid \hat{p} \mid m\rangle =$

$$\int_0^a dx' \int_0^a dx'' \langle n|x'\rangle \langle x'|\hat{p}|x''\rangle \langle x''|m\rangle = \int_0^a x'\,dx' \int_0^a x''\,dx'' \langle n|x'\rangle \frac{\hbar}{i}\frac{d}{dx'}\delta(x'-x'')\langle x''|m\rangle =$$

$$\int_0^a dx' \langle n|x'\rangle \frac{\hbar}{i}\frac{d}{dx'}\langle x'|m\rangle = \frac{2}{a}\frac{\hbar}{i}\frac{m\pi}{a}\int_0^a dx' \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)$$

$$\text{(for } m = n) \quad = \frac{2}{a}\frac{\hbar}{i}\frac{m\pi}{a}\frac{1}{2\left(\frac{n\pi}{a}\right)}\sin^2\left(\frac{n\pi x}{a}\right)\{x, 0, a\} = 0$$

$$(\text{for } m \neq n)\ \frac{2}{a}\frac{\hbar}{i}\,m\left\{\frac{\cos\left[(m-n)\frac{\pi x}{a}\right]}{2(m-n)} - \frac{\cos\left[(m+n)\frac{\pi x}{a}\right]}{2(m+n)}\right\}\{x, 0, a\}$$

$$= \frac{2}{a}\frac{\hbar}{i}\,m\left\{\frac{(-1)^{(m-n)}-1}{2(m-n)} - \frac{(-1)^{m+n}-1}{2(m+n)}\right\} =$$

$$\frac{1}{a}\frac{\hbar}{i}\,m\left\{\frac{(-1)^{(m+n)}-1}{(m-n)} - \frac{(-1)^{m+n}-1}{(m+n)}\right\} =$$

$$= \frac{1}{a}\frac{\hbar}{i}\,m[(-1)^{(m+n)}-1]\left\{\frac{1}{m-n} - \frac{1}{m+n}\right\} =$$

$$= \frac{1}{a}\frac{\hbar}{i}\,m[(-1)^{(m+n)}-1]\left\{\frac{m+n-m+n}{(m-n)(m+n)}\right\} =$$

$$\frac{1}{a}\frac{\hbar}{i}[(-1)^{(m+n)}-1]\left(\frac{mn}{m^2-n^2}\right)$$

$$(\text{if } m+n \text{ is even}) = 0$$

$$(\text{if } m+n \text{ is odd})\ \frac{2}{a}\frac{\hbar}{i}\left(\frac{nm}{n^2-m^2}\right)$$

$$\begin{pmatrix} 0 & \frac{-2}{3} & 0 & \frac{-4}{16} & 0 & \square \\ \frac{2}{3} & 0 & \frac{6}{-5} & 0 & \frac{10}{-21} & \square \\ 0 & \frac{6}{-5} & 0 & \frac{12}{-7} & 0 & \square \\ \frac{4}{16} & 0 & \frac{12}{7} & 0 & \frac{20}{-9} & 0 \\ 0 & \frac{10}{21} & 0 & \frac{20}{9} & 0 & \square \\ \square & \square & \square & 0 & \square & \square \end{pmatrix}\frac{2}{a}\frac{\hbar}{i}$$

```
n = 4
m = 5
Integrate[Sin[(n * Pi)/a * x] * Cos[(m * Pi)/a * x] * m * Pi / a, {x, 0, a}]
```

4

5

$$-\frac{40}{9}$$

$$\begin{pmatrix} 0 & \frac{-4}{3} & 0 & \frac{-8}{15} & 0 & \frac{-12}{35} \\ \frac{4}{3} & 0 & \frac{-12}{5} & 0 & \frac{-20}{21} & 0 \\ 0 & \frac{12}{5} & 0 & \frac{-24}{7} & 0 & \square \\ \frac{-8}{16} & 0 & \frac{24}{7} & 0 & \frac{-40}{9} & 0 \\ 0 & \frac{20}{21} & 0 & \frac{40}{9} & 0 & \square \\ \frac{12}{35} & 0 & \square & 0 & \square & 0 \end{pmatrix} \frac{2}{a} \frac{\hbar}{i}$$

note : $\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$ fron $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

for x the matrix elements are $\langle n \mid \hat{x} \mid m\rangle$ where $\langle x \mid n\rangle = \sqrt{\frac{2}{a}}\,\sin\left(\frac{n\pi x}{a}\right)$

$\langle n \mid \hat{x} \mid m\rangle =$

$$\int_0^a dx' \int_0^a dx'' \langle n|x'\rangle \langle x'|\hat{x}|x''\rangle \langle x''|m\rangle = \int_0^a dx' \int_0^a x''\, dx'' \langle n|x'\rangle \delta(x'-x'') \langle x''|m\rangle =$$

$$\int_0^a x'\, dx' \langle n|x'\rangle \langle x'|m\rangle = \frac{2}{a}\int_0^a dx'\, x' \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) =$$

(for m = n) $= \frac{a}{2}$

$$= \frac{2}{a}\int_0^a dx'\, x'\left(\frac{1}{2}\cos(n-m)\frac{\pi x}{a} - \frac{1}{2}\cos(n+m)\frac{\pi x}{a}\right) =$$

$$\frac{1}{a}\int_0^a dx'\, x' \cos(n-m)\frac{\pi x}{a} - \frac{1}{2a}\int_0^a dx'\, x' \cos(n+m)\frac{\pi x}{a} =$$

$$\frac{1}{a}\Big[\frac{1}{\left((n-m)\frac{\pi}{a}\right)^2}\cos(n-m)\frac{\pi x}{a} + \frac{x}{(n-m)\frac{\pi}{a}}\sin(n-m)\frac{\pi x}{a} -$$

$$\frac{1}{\left((n+m)\frac{\pi}{a}\right)^2}\cos(n+m)\frac{\pi x}{a} - \frac{x}{(n+m)\frac{\pi}{a}}\sin(n+m)\frac{\pi x}{a}\Big]$$

$$= \frac{1}{a}\Big[\frac{1}{\left((n-m)\frac{\pi}{a}\right)^2}\cos(n-m)\pi - \frac{1}{\left((n+m)\frac{\pi}{a}\right)^2}\cos(n+m)\pi -$$

$$\frac{1}{\left((n-m)\frac{\pi}{a}\right)^2} + \frac{1}{\left((n+m)\frac{\pi}{a}\right)^2}\Big]$$

$(n+m \text{ even}) \quad = 0$

$$(n+m \text{ odd}) = \frac{2}{a}\Big[\frac{1}{\left((n+m)\frac{\pi}{a}\right)^2} - \frac{1}{\left((n-m)\frac{\pi}{a}\right)^2}\Big] =$$

$$\frac{2a}{\pi^2}\Big[\frac{(n-m)^2-(n+m)^2}{(n+m)^2(n-m)^2}\Big] = \frac{2a}{\pi^2}\Big[\frac{-4mn}{(n+m)^2(n-m)^2}\Big]$$

$$\begin{pmatrix} \frac{\pi^2}{4} & \frac{-8}{9} & 0 & \frac{-16}{255} & 0 & \square \\ \frac{-8}{9} & \frac{\pi^2}{4} & \frac{-24}{25} & 0 & \frac{-40}{441} & 0 \\ 0 & \frac{-24}{25} & \frac{\pi^2}{4} & \frac{-48}{49} & 0 & \square \\ \frac{-16}{255} & 0 & \frac{-48}{49} & \frac{\pi^2}{4} & \frac{-80}{81} & 0 \\ 0 & \frac{-40}{441} & 0 & \frac{-80}{81} & \frac{\pi^2}{4} & \square \\ \square & 0 & \square & 0 & \square & \square \end{pmatrix} \frac{2a}{\pi^2}$$

```
m =.
n =.
Integrate[x * Sin[(n * Pi)/a * x] * Sin[(m * Pi)/a * x], {x, 0, a}]
```

$$\frac{\cos((m-n)\pi)\,a^2}{2\,(m-n)^2\,\pi^2}-\frac{\cos((m+n)\pi)\,a^2}{2\,(m+n)^2\,\pi^2}-\frac{n\sin((m-n)\pi)\,a^2}{2\,(m-n)^2\,\pi}+\frac{m\sin((m-n)\pi)\,a^2}{2\,(m-n)^2\,\pi}-\frac{m\sin((m+n)\pi)\,a^2}{2\,(m+n)^2\,\pi}-\frac{n\sin((m+n)\pi)\,a^2}{2\,(m+n)^2\,\pi}-\frac{a^2}{2\,(m-n)^2\,\pi^2}+\frac{a^2}{2\,(m+n)^2\,\pi^2}$$

$$\left(\frac{\cos((m-n)\pi)-1}{2\,(m-n)^2}-\frac{\cos((m+n)\pi)-1}{2\,(m+n)^2}\right)\frac{2}{a}\,\frac{a^2}{\pi^2}$$

for $n+m$ odd

$$\frac{-2}{a}\,\frac{a^2}{\pi^2}\left(\frac{1}{(m-n)^2}-\frac{1}{(m+n)^2}\right)=\frac{-2}{a}\,\frac{a^2}{\pi^2}\left(\frac{(m+n)^2-(m-n)^2}{(m-n)^2\,(m+n)^2}\right)=\frac{2}{a}\,\frac{a^2}{\pi^2}\left(\frac{-4\,\mathrm{mn}}{(m-n)^2\,(m+n)^2}\right)$$

```
m = 1
n = 2
Integrate[x * Sin[(n * Pi)/a * x] * Sin[(m * Pi)/a * x], {x, 0, a}]
```

1

2

$$-\frac{8\,a^2}{9\,\pi^2}$$

$$\begin{pmatrix} \frac{\pi^2}{4} & \frac{-8}{9} & 0 & \frac{-16}{255} & 0 & \square \\ \frac{-8}{9} & \frac{\pi^2}{4} & \frac{-24}{25} & 0 & \frac{-40}{441} & 0 \\ 0 & \frac{-24}{25} & \frac{\pi^2}{4} & \frac{-48}{49} & 0 & \square \\ \frac{-16}{255} & 0 & \frac{-48}{49} & \frac{\pi^2}{4} & \frac{-80}{81} & 0 \\ 0 & \frac{-40}{441} & 0 & \frac{-80}{81} & \frac{\pi^2}{4} & \square \\ \square & 0 & \square & 0 & \square & \square \end{pmatrix}\frac{2}{a}\,\frac{a^2}{\pi^2}$$

```
psi[x_] = Sqrt[2 / a] * Sin[3 * Pi * x / a]
```

$$\sqrt{2}\,\sqrt{\frac{1}{a}}\,\sin\left(\frac{3\,\pi\,x}{a}\right)$$

```
phi[k_] = Integrate[Exp[I * k * x] * psi[x] / Sqrt[2 * Pi], {x, 0, a}]
```

$$-\frac{3\,(1+e^{i\,a\,k})\,\sqrt{\pi}}{\sqrt{\frac{1}{a}}\,(a^2\,k^2-9\,\pi^2)}$$

```
Integrate[x^2 * psi[x] * psi[x], x]
```

$$\frac{36\,\pi^3\,x^3-6\,a^2\,\pi\cos\left(\frac{6\,\pi\,x}{a}\right)x+(a^3-18\,a\,\pi^2\,x^2)\sin\left(\frac{6\,\pi\,x}{a}\right)}{108\,a\,\pi^3}$$

```
xav = Integrate[x * psi[x] * psi[x], {x, 0, a}]
```

$$\frac{a}{2}$$

```
xxav = Integrate[x * x * psi[x] * psi[x], {x, 0, a}]
```

$$\frac{a^2}{3}-\frac{a^2}{18\,\pi^2}$$

```
ppav = Integrate[psi[x] * psi[x], {x, 0, a}]
```

1

$$\mathbf{dx} = \frac{a^2}{3} - \frac{a^2}{18\pi^2} - \frac{a^2}{4} \qquad a\sqrt{\frac{1}{12} - \frac{1}{18\pi^2}} \qquad 6\pi\sqrt{\frac{1}{12} - \frac{1}{18\pi^2}}\ \frac{\hbar}{2}$$

$$\mathbf{dp} = 9\,\frac{\pi^2}{a^2}\,\hbar^2 \qquad 3\,\frac{\pi\hbar}{a}$$

$$\mathbf{dxdp} = 9\pi^2\hbar^2\left(\frac{1}{3} - \frac{1}{4} - \frac{1}{18\pi^2}\right) = 9\pi^2\hbar^2\left(\frac{1}{12} - \frac{1}{18\pi^2}\right) = \left(\pi^2\,\frac{9}{12} - \frac{1}{2}\right)\hbar^2 = (\pi^2\,3 - 2)\,\frac{\hbar^2}{4}$$

Check and see what the answer is for Δp if we do it in momentum space

$$\mathbf{phip[k_]} := \frac{-3*(1+\mathrm{Exp}[I*k*a])\sqrt{\pi a}}{k^2a^2 - 9\pi^2}$$

$$\mathbf{phipc[k_]} := \frac{-3*(1+\mathrm{Exp}[-I*k*a])\sqrt{\pi a}}{k^2a^2 - 9\pi^2}$$

$$\mathbf{pp[k_]} := \frac{18*\pi*a*(1+1*\mathrm{Cos}[k*a])}{(k^2*a^2 - 9\pi^2)^2}$$

pp[*k*]

$$\frac{18\,a\,\pi\,(\cos(a\,k)+1)}{(a^2\,k^2 - 9\,\pi^2)^2}$$

kav = Integrate[pp[*k*] ∗ *k*, {*k*, −∞, ∞}]

0

kkav = Integrate[pp[*k*] ∗ *k* ∗ *k*, {*k*, −∞, ∞}]

$$\mathrm{If}\left[\mathrm{Im}(a) == 0,\ -\frac{9\,\pi^2\,\mathrm{sgn}(a)}{a^2},\ \mathrm{Integrate}\left[\frac{18\,a\,\pi\cos(a\,k)\,k^2}{(a^2\,k^2 - 9\,\pi^2)^2} + \frac{18\,a\,\pi\,k^2}{(a^2\,k^2 - 9\,\pi^2)^2},\ \{k, -\infty, \infty\},\ \mathrm{Assumptions} \to \mathrm{Im}(a) \neq 0\right]\right]$$

kkav $*\,\hbar^{2=}\ \dfrac{9\pi^2\hbar^2}{a^2} = \hbar^2\left(\dfrac{3\pi}{a}\right)^2$ **which is the same as we got above so its OK. (is there *a* sign problem?)**

```
m = 4
n = 3
Integrate[Sin[(n * Pi)/a * x] * Cos[(m * Pi)/a * x] * m * Pi / a, {x, 0, a}]
```

4

3

$-\frac{24}{7}$

```
I
```

```
m = 1
n = 3
Integrate[x * Sin[(n * Pi)/a * x] * Sin[(m * Pi)/a * x], {x, 0, a}]
```

1

3

0

2) Let $\hat{x}$ and $\hat{p}$ be the quantum mechanical operators for position and momentum (1-dimensional)

a) Evaluate the comutator $[\hat{p}, e^{\frac{\mathrm{i}a\hat{x}}{\hbar}}]$

Note : $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ **and we know** $[x, p] = i\hbar$

so $[p, x^1] = -i\hbar$, $[p, x^2] = x[p, x] + [p, x]\, x = -2\, i\hbar x$

$[p, x^3] = x^2[p, x] + [p, x^2]\, x = -i\hbar x^2 - 2\, i\hbar x^2 = -3\, i\hbar x^2$ **and finally** $[p, x^n] = -n(i\hbar)\, x^{n-1}$

Also note $[p, x^0] = 0$

$$\left[p, e^{\frac{ia\hat{x}}{\hbar}}\right] = \left[p, \sum_{n=0}^{\infty} \frac{i^n a^n x^n}{\hbar^n n!}\right] = \sum_{n=1}^{\infty} \frac{i^n a^n}{\hbar^n n!}[p, x^n] = -\sum_{n=1}^{\infty} \frac{i^n a^n}{\hbar^n n!}(i\hbar)\, n x^{n-1} =$$

$$-\sum_{n=1}^{\infty} \frac{i^2 i^{n-1} a a^{n-1}}{\hbar^{n-1}(n-1)!} x^{n-1} = a \sum_{n=1}^{\infty} \frac{i^{n-1} a^{n-1}}{\hbar^{n-1}(n-1)!} x^{n-1} = a \sum_{n=0}^{\infty} \frac{i^n a^n}{\hbar^n n!} x^n = a e^{\frac{ia\hat{x}}{\hbar}}$$

Some folks wanted to do it in the x representation. This works, but you have to be careful. First we will let the commutator hit a ket

$[\hat{p}, e^{\frac{ia\hat{x}}{\hbar}}]|\alpha\rangle = \hat{p}\, e^{\frac{ia\hat{x}}{\hbar}}|\alpha\rangle - e^{\frac{ia\hat{x}}{\hbar}}\hat{p}|\alpha\rangle$=A-B

A=$\hat{p}\, e^{\frac{ia\hat{x}}{\hbar}}|\alpha\rangle = \int dx' \int dx''' \int dx'' |x''\rangle\langle x''|\hat{p}|x'''\rangle\langle x'''|\, e^{\frac{ia\hat{x}}{\hbar}}|x'\rangle\langle x'|\alpha\rangle = \int dx' \int dx''' \int dx'' |x''\rangle\langle x''|\hat{p}|x'''\rangle\langle x'''|e^{\frac{iax'}{\hbar}}|x'\rangle\langle x'|\alpha\rangle =$

$= \int dx' \int dx'' |x''\rangle\langle x''|\, \hat{p}\, |x'\rangle e^{\frac{iax'}{\hbar}} \langle x'|\alpha\rangle = \frac{\hbar}{i}\int dx' |x'\rangle \frac{d}{dx'} e^{\frac{iax'}{\hbar}} \langle x'|\alpha\rangle = \frac{\hbar}{i}\int dx' |x'\rangle \frac{d}{dx'} e^{\frac{iax'}{\hbar}} \langle x'|\alpha\rangle =$

$\frac{\hbar}{i}\int dx' |x'\rangle (e^{\frac{iax'}{\hbar}} \frac{d}{dx'} \langle x'|\alpha\rangle + \langle x'|\alpha\rangle \frac{ia'}{\hbar} e^{\frac{iax'}{\hbar}}) = \frac{\hbar}{i}\int dx' |x'\rangle e^{\frac{iax'}{\hbar}} \frac{d}{dx'} \langle x'|\alpha\rangle + \frac{\hbar}{i}\int dx' |x'\rangle \langle x'|\alpha\rangle \frac{ia}{\hbar} e^{\frac{iax'}{\hbar}}$

B= $e^{\frac{ia\hat{x}}{\hbar}}\hat{p}|\alpha\rangle = \int dx' \int dx''' \int dx'' |x'''\rangle\langle x'''|\, e^{\frac{ia\hat{x}}{\hbar}}|x''\rangle\langle x''|\hat{p}|x'\rangle\langle x'|\alpha\rangle = \int dx' \int dx'' |x''\rangle\, e^{\frac{iax''}{\hbar}} \langle x''|\hat{p}|x'\rangle\langle x'|\alpha\rangle =$

$\frac{\hbar}{i}\int dx' |x'\rangle e^{\frac{iax'}{\hbar}} \frac{d}{dx'} \langle x'|\alpha\rangle$

A-B= $\frac{\hbar}{i}\int dx' |x'\rangle \langle x'|\alpha\rangle \frac{ia}{\hbar} e^{\frac{iax'}{\hbar}} = \int dx'\ a\ e^{\frac{iax'}{\hbar}} |x'\rangle\langle x'|\alpha\rangle = \int dx'\ a\ e^{\frac{ia\hat{x}}{\hbar}} |x'\rangle\langle x'|\alpha\rangle = a e^{\frac{ia\hat{x}}{\hbar}} |\alpha\rangle$ So

$[\hat{p}, e^{\frac{ia\hat{x}}{\hbar}}] = a e^{\frac{ia\hat{x}}{\hbar}}$

b) Using this result evaluate $e^{\frac{ia\hat{x}}{\hbar}}|p'\rangle$. What is the significance of the operator $e^{\frac{ia\hat{x}}{\hbar}}$? (Hint: find $\hat{p} e^{\frac{ia\hat{x}}{\hbar}}|p'\rangle$)

$$\left[p, e^{\frac{ia\hat{x}}{\hbar}}\right] = a e^{\frac{ia\hat{x}}{\hbar}}$$

$$\left[p, e^{\frac{ia\hat{x}}{\hbar}}\right] | p' \rangle = \hat{p}\, e^{\frac{ia\hat{x}}{\hbar}} |p'\rangle - e^{\frac{ia\hat{x}}{\hbar}} \hat{p} |p'\rangle = a e^{\frac{ia\hat{x}}{\hbar}} | p' \rangle$$

$$\hat{p}\, e^{\frac{ia\hat{x}}{\hbar}} |p'\rangle = e^{\frac{ia\hat{x}}{\hbar}} \hat{p} |p'\rangle + a e^{\frac{ia\hat{x}}{\hbar}} | p' \rangle = e^{\frac{ia\hat{x}}{\hbar}} p' |p'\rangle + a e^{\frac{ia\hat{x}}{\hbar}} | p' \rangle = (p' + a)\, e^{\frac{ia\hat{x}}{\hbar}} | p' \rangle$$

So $e^{\frac{ia\hat{x}}{\hbar}} | p' \rangle$ **is an eigenfunction of** $\hat{p}$ **with eigenvalue** $p' + a$, **so the operator** $e^{\frac{ia\hat{x}}{\hbar}}$ **is** a **momentum boost**

3) A certain observable in quantum mechanics has a matrix representation as follows:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

THIS WAS A TYPO IN THE PROBLEM!!! I WILL GRADE IT ACCORDINGLY IT SHOULD HAVE BEEN (note that the other problem is solvable but a real pain)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Find the normalized eigenvectors and corresponding eigenvalues.

Is there a degeneracy?

[It turns out that this observable will be the spin for a spin-1 particle]

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \mathbf{det}\begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix} = 0 \qquad -\lambda^3 + \lambda = 0 \qquad \lambda = 0, 1, -1$$

eigenkets for λ=0

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad \mathbf{b=0} \quad \mathbf{a+c=0} \quad \mathbf{b=0} \qquad \text{So } \frac{1}{\sqrt{2}}(1,0,-1) \quad \text{- no degeneracy}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \pm\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \frac{1}{\sqrt{2}}\mathbf{b=\pm a} \qquad \frac{1}{\sqrt{2}}(a+c)=\pm b \quad \frac{1}{\sqrt{2}}b=\pm c \qquad \text{so } a=c$$

for λ=1 let a=1 then b=$\sqrt{2}$ and c= 1 **normalizing** we get $\frac{1}{\sqrt{4}}(1,\sqrt{2},1) = (\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$

for λ=-1 let a=1 then b=-$\sqrt{2}$ and c= 1 normalizing we get $\frac{1}{\sqrt{4}}(1,-\sqrt{2},1) = (\frac{1}{2}, \frac{-1}{\sqrt{2}}, \frac{1}{2})$

$$a = \frac{1}{\sqrt{2}}\{\{0, 1, 0\}, \{1, 0, 1\}, \{0, 1, 0\}\}$$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Eigenvalues[*a*]

$\{-1, 1, 0\}$

Eigenvectors[*a*]

$$\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ 1 & \sqrt{2} & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

4) A spin $\frac{1}{2}$ system is known to be in an eigenstate of $\hat{\vec{S}}\cdot\vec{n}$ with eigenvalue $\frac{+\hbar}{2}$ where $\vec{n}$ is a unit vector lying on the x-z plane that makes an angle γ with the positive z axis. In each of these questions, check for the special cases where $\gamma = 0, \frac{\pi}{2}, \pi$

a) find the eigenkets of $\hat{\vec{S}}\cdot\vec{n}$ for this case.

First we know $n_y = 0$ so $n = (\sin\gamma,\ 0,\ \cos\gamma)$ so

$$\hat{\vec{S}}\cdot\vec{n} = \frac{\hbar}{2}\sin\gamma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\cos\gamma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix}$$

The eigenvalues of this thing are $\pm\frac{\hbar}{2}$ so we look for eigenkets

$$\begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix} \quad a\cos\gamma + b\sin\gamma = \pm a \quad \text{and } a\sin\gamma - b\cos\gamma = \pm b \quad \text{take } b = 1$$

a =

$$\frac{\cos\gamma \pm 1}{\sin\gamma} \quad \text{and normalizing we get } \frac{1}{\sqrt{\left(\frac{\cos\gamma\pm 1}{\sin\gamma}\right)^2 + 1}}\begin{pmatrix} \frac{\cos\gamma\pm 1}{\sin\gamma} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{(\cos\gamma \pm 1)^2 + \sin^2\gamma}}\begin{pmatrix} \cos\gamma \pm 1 \\ \sin\gamma \end{pmatrix} =$$

$$\frac{1}{\sqrt{\cos^2\gamma \pm 2\cos\gamma + 1 + \sin^2\gamma}}\begin{pmatrix} \cos\gamma \pm 1 \\ \sin\gamma \end{pmatrix} = \frac{1}{\sqrt{2 \pm 2\cos\gamma}}\begin{pmatrix} \cos\gamma \pm 1 \\ \sin\gamma \end{pmatrix}$$

This actually simplifies – but you didnt need to do it

$$\frac{1}{\sqrt{2 \pm 2\cos\gamma}}\begin{pmatrix} \cos\gamma \pm 1 \\ \sin\gamma \end{pmatrix} = \begin{pmatrix} \frac{\pm(1\pm\cos\gamma)}{\sqrt{2(1\pm\cos\gamma)}} \\ \frac{\sin\gamma}{\sqrt{2(1\pm\cos\gamma)}} \end{pmatrix} = \begin{pmatrix} \frac{\pm\sqrt{(1\pm\cos\gamma)}}{\sqrt{2}} \\ \frac{\sin\gamma}{\sqrt{2(1\pm\cos\gamma)}} \end{pmatrix}$$

$$(\text{for } +)\ \begin{pmatrix} \cos\frac{\gamma}{2} \\ \frac{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{2\cos\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix} \qquad (\text{for } -)\ \begin{pmatrix} -\sin\frac{\gamma}{2} \\ \frac{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{2\sin\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} -\sin\frac{\gamma}{2} \\ \cos\frac{\gamma}{2} \end{pmatrix}$$

Null

b) suppose $\hat{S}_x$ is measured. What is the probability of getting $\frac{+\hbar}{2}$?

Since it is in an eigenstate of $\hat{\vec{S}} \cdot \vec{n}$ with eigenvalue $\frac{+\hbar}{2}$ it is $\begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix}$

So we begin with the eigenstate $|S \cdot n; +\rangle$,

and ask what the probability is that it is $S_x +$, that is it will yield *a* value of $+\frac{\hbar}{2}$ when measureing S_x

$$P = \left| \langle S_{x;+} | S \cdot n; + \rangle \right|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix} \right|^2 =$$

$$\frac{1}{2}\left(\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\right)^2 = \frac{1}{2}\left(\cos^2\frac{\gamma}{2} + \sin^2\frac{\gamma}{2} + 2\cos\frac{\gamma}{2}\sin\frac{\gamma}{2}\right) = \frac{1}{2}(1 + \sin\gamma)$$

Lets check and see if this makes sense for $\gamma = 0, \frac{\pi}{2}, \pi$.

For $\gamma = 0, \pi$ its either along or antiparallel to the z axis and $\vec{S} \cdot \vec{n}$ is just S_z so we would expect that the probability to measure $\frac{+\hbar}{2}$ is $\frac{1}{2}$ which is what we get. When $\gamma = \frac{\pi}{2}$ it points in the x direction so the probabililty should be 1 which it is. Good.

Another way

$$\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle =$$

$$\cos\frac{\gamma}{2}\left(\frac{1}{\sqrt{2}}|S_x +\rangle + \frac{1}{\sqrt{2}}|S_x -\rangle\right) + \sin\frac{\gamma}{2}\left(\frac{1}{\sqrt{2}}|S_x +\rangle - \frac{1}{\sqrt{2}}|S_x -\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\left(\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\right)|S_x +\rangle + \frac{1}{\sqrt{2}}\left(\cos\frac{\gamma}{2}|+\rangle - \sin\frac{\gamma}{2}\right)|S_x -\rangle$$

so the probability to measure $\frac{+\hbar}{2}$ when S_x is measured is $\frac{1}{2}\left(\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\right)^2$

$$= \frac{1}{2}\left(\cos^2\frac{\gamma}{2} + \sin^2\frac{\gamma}{2} + 2\cos\frac{\gamma}{2}\sin\frac{\gamma}{2}\right) = \frac{1}{2}(1 + \sin\gamma)$$

Null

c) Evaluate $\Delta \hat{S}_x$.

$$\langle S_x\rangle = \left(\cos\frac{\gamma}{2} \quad \sin\frac{\gamma}{2}\right)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\gamma}{2}\\ \sin\frac{\gamma}{2}\end{pmatrix} = \frac{\hbar}{2}\left(\cos\frac{\gamma}{2} \quad \sin\frac{\gamma}{2}\right)\begin{pmatrix}\sin\frac{\gamma}{2}\\ \cos\frac{\gamma}{2}\end{pmatrix} = \frac{\hbar}{2}\,2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2} = \frac{\hbar}{2}\sin\gamma$$

$$\langle S^2{}_x\rangle = \left(\cos\frac{\gamma}{2} \quad \sin\frac{\gamma}{2}\right)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\gamma}{2}\\ \sin\frac{\gamma}{2}\end{pmatrix} =$$

$$\frac{\hbar^2}{4}\left(\cos\frac{\gamma}{2} \quad \sin\frac{\gamma}{2}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sin\frac{\gamma}{2}\\ \cos\frac{\gamma}{2}\end{pmatrix} = \frac{\hbar^2}{4}\left(\cos\frac{\gamma}{2} \quad \sin\frac{\gamma}{2}\right)\begin{pmatrix}\cos\frac{\gamma}{2}\\ \sin\frac{\gamma}{2}\end{pmatrix} = \frac{\hbar^2}{4}$$

$$\Delta S_x = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2\gamma} = \frac{\hbar}{2}\cos\gamma$$

If $\gamma = \frac{\pi}{2}$ it is an eigenstate of S_x so the uncertainty is zero. It is. Good

If $\gamma = 0, \pi$ then there should be an uncertainty $-$ it is $\frac{\hbar}{2}$, good.

0