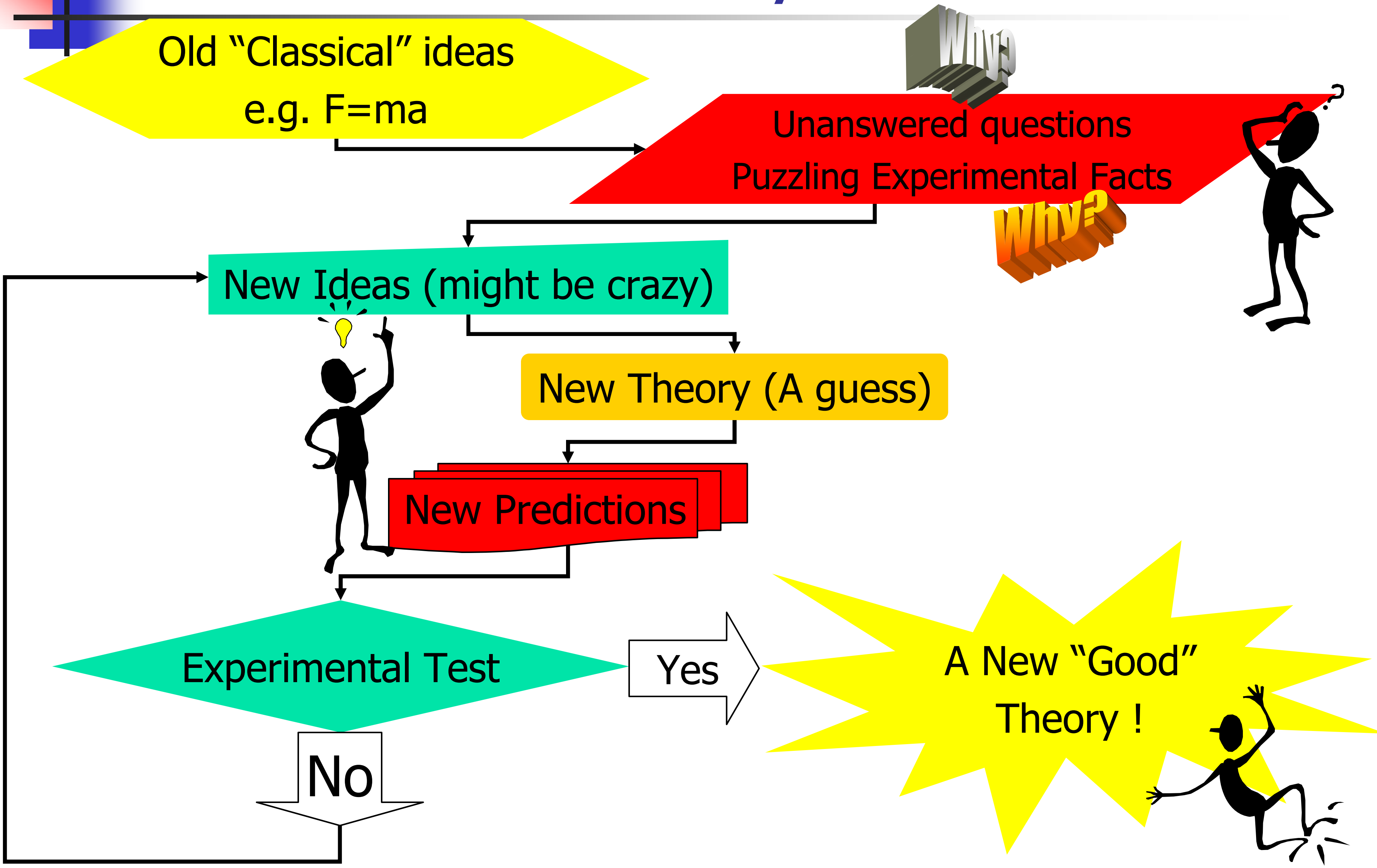
Physics 165

Notes on field theory, symmetries, QED, symmetry breaking, and the E-W theory

Richard Seto

Oct 2008

How does a theory come to be?



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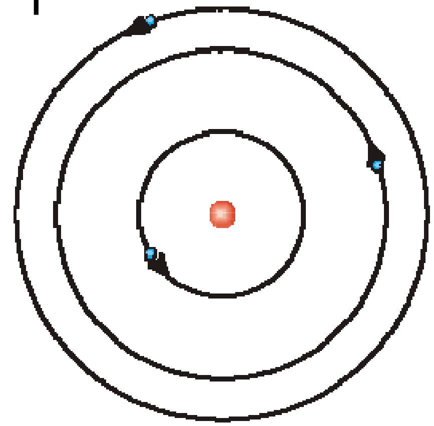
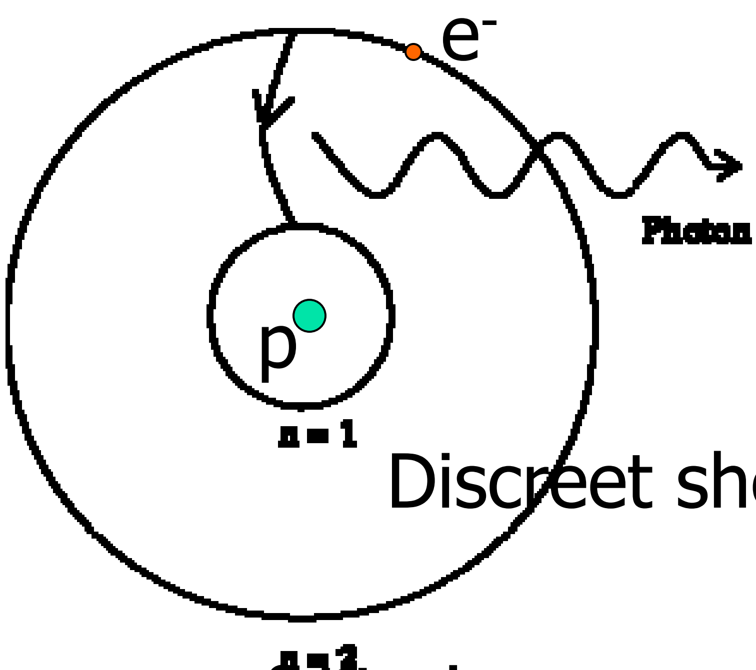
"Once upon a time"

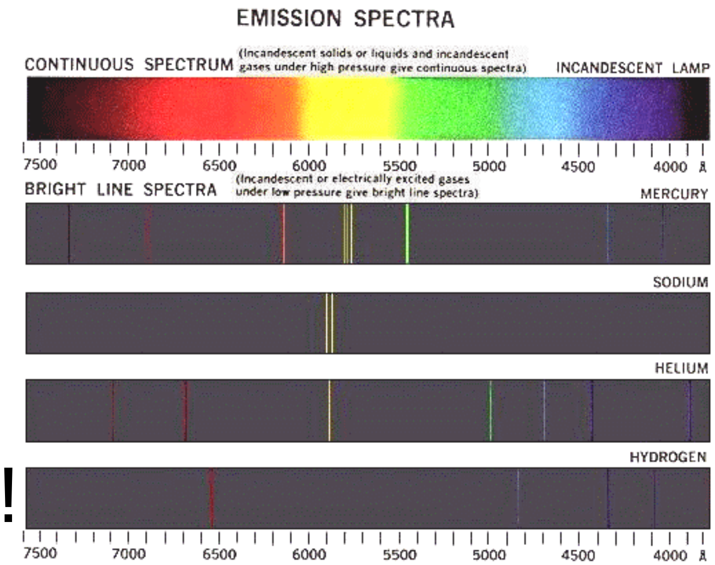
- Pre-1900 "The standard model"
  - Newton, Kepler $\vec{x}, \vec{p}, E, \vec{v}, \vec{a}, \vec{F}=m\vec{a}, \vec{F}=\frac{GMm}{r^2}$
  - Maxwell $B = \nabla \times A,\ E = -\frac{1}{c}\frac{\partial A}{\partial t}$
  - Pretty good
    - Explained orbits of planets
    - Explained static electricity
  - Maxwell –"We have a complete theory. Physics will be over in 10 years…"
    - But there were a few minor problems



The bad guys



Discreet shells!



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- Spectra of Hydrogen atom
- Strange – like a merry-go-round which only went v=1mph, 5mph, 10mph,….
- THE WHITE KNIGHT – QUANTUM MECHANICS

Getting to QM

quantum mechanics had been discovered in 1926 in two rather distinct forms by Erwin Schrödinger and Werner Heisenberg. It was von Neumann's insight that the natural language of quantum mechanics was that of self-adjoint operators on Hilbert space.

- Classically

$$\mathrm{E}=\frac{\mathrm{p}^2}{2\mathrm{m}}+\mathrm{V}(r)$$

- To "quantize"
  - Particles change to "wave functions"
  - E,p,V Change to operators (E→H, p→p, V→V)
  - But operate on what??
  - Answer: ψ(x)
    - the "wave function"
  - ψ is the "real thing"

Guess: $\boldsymbol{H}\psi(x,t)=\left[\frac{\boldsymbol{p}^2}{2m}+\boldsymbol{V}(r)\right]\psi(x,t)$

Formally quantize: $[\boldsymbol{A},\boldsymbol{B}]\psi=(\boldsymbol{AB}-\boldsymbol{BA})\psi$

$\rightarrow[\boldsymbol{x},\boldsymbol{p}]\psi=i\hbar\psi$ where $\hbar$ is very small

(idea from $\Delta x\Delta p\geq\hbar$ where classically $\hbar$=0)

One Solution: x $\rightarrow\boldsymbol{x}=$ x p $\rightarrow\boldsymbol{p}=\frac{\hbar}{i}\frac{\partial}{\partial x}$

check it!

$$[\boldsymbol{x},\boldsymbol{p}]\psi=\boldsymbol{x}\frac{\hbar}{i}\frac{\partial}{\partial x}(\psi)-\frac{\hbar}{i}\frac{\partial}{\partial x}(\boldsymbol{x}\psi)$$

$$=x\frac{\hbar}{i}\frac{\partial}{\partial x}(\psi)-x\frac{\hbar}{i}\frac{\partial}{\partial x}(\psi)-\frac{\hbar}{i}\left(\frac{\partial}{\partial x}x\right)\psi$$

$=i\hbar\psi$ It works!

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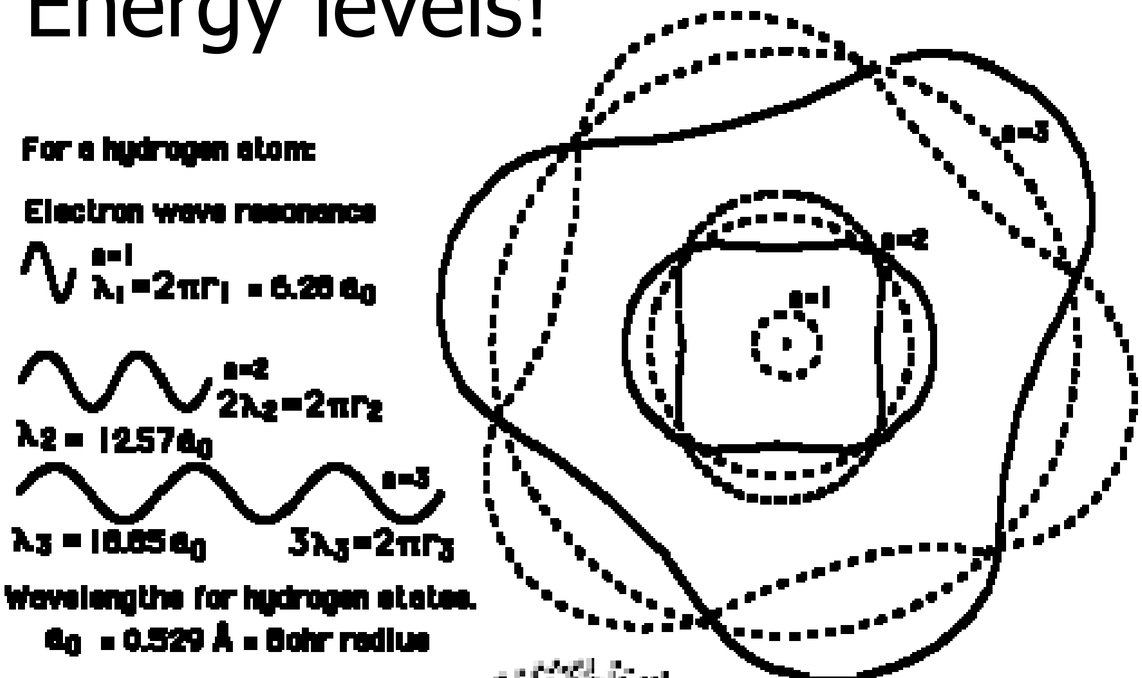
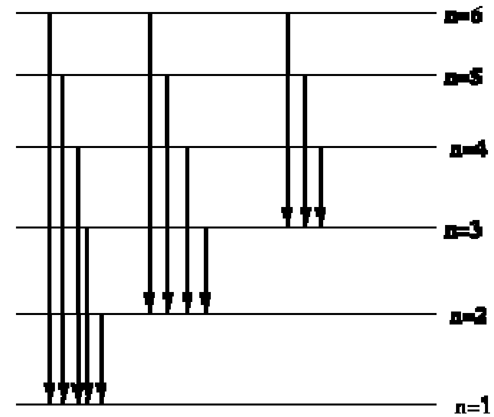
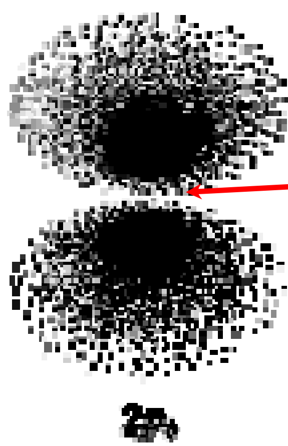
What does it predict?

- Energy levels!





- BUT!



Zero probability!

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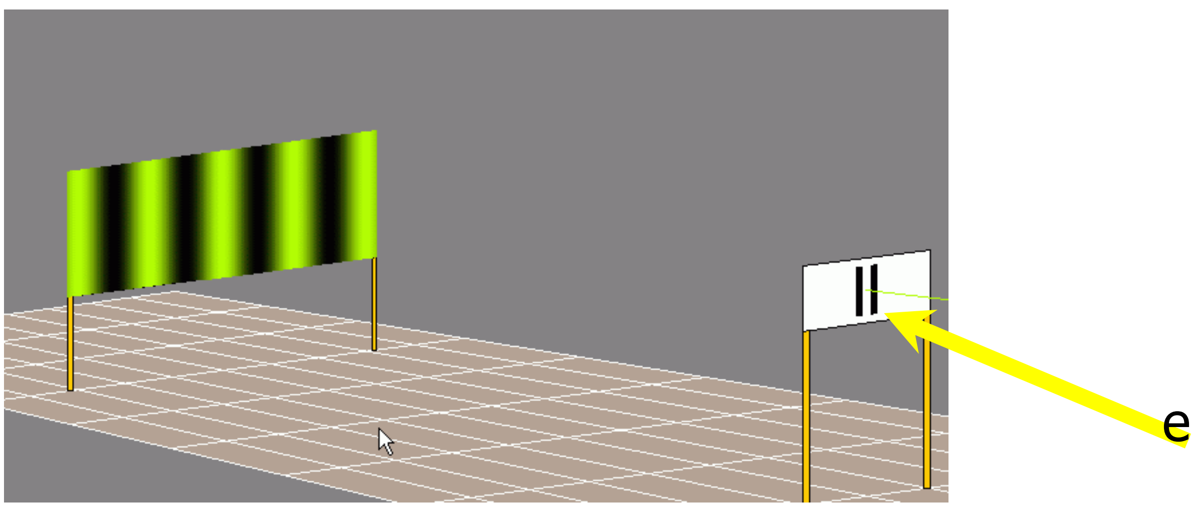
What is ψ ?

- Answer $\psi^*\psi$ is a probability
- We have lost the normal notion of a particle with position and velocity
- Crazy, but it EXPERIMENT tells us it works
- E.g. double slit experiment with electrons

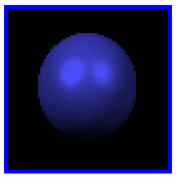


wave function of H atom

$$H\psi(x,t)=\left[\frac{p^2}{2m}+V(r)\right]\psi(x,t)$$

$$\mathrm{x}\to\vec{r}\quad \mathrm{p}\to\boldsymbol{p}=\frac{\hbar}{i}\vec{\nabla}\quad V(r)=\frac{e^2}{r^2}\hat{r}$$

- **n=1:**

	m=0
l=0	

- **n=2:**

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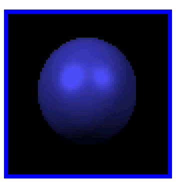
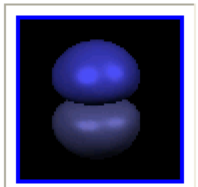
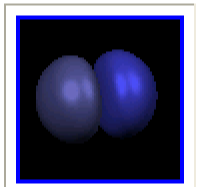
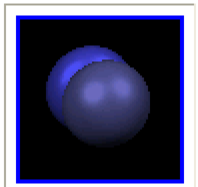
l=0			
l=1			

- **n=3:**

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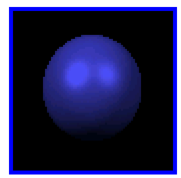
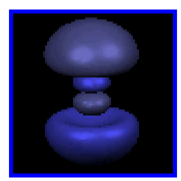
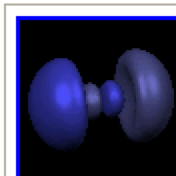
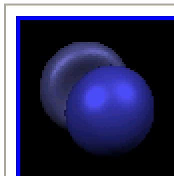
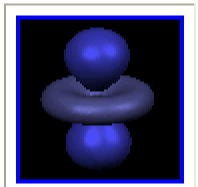
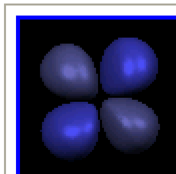
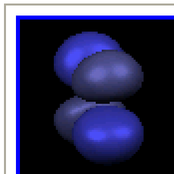
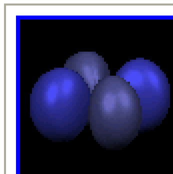
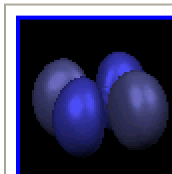
l=0					
l=1					
l=2					

- **n=4:**

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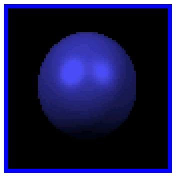
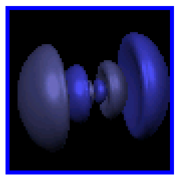
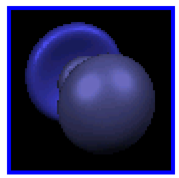
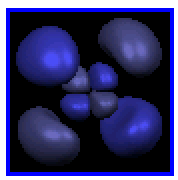
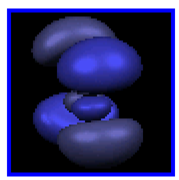
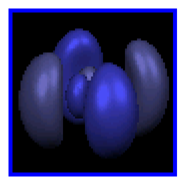
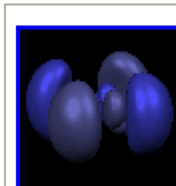
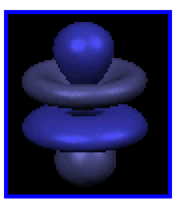
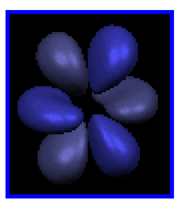
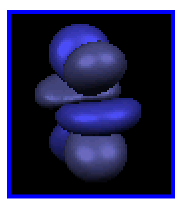
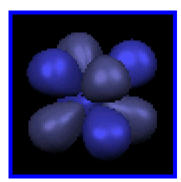
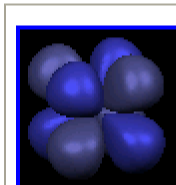
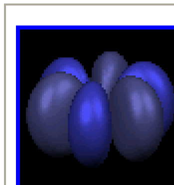
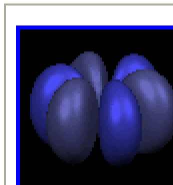
l=0							
l=1							
l=2							
l=3							

Happiness again?

- Dirac – "underlying physical laws necessary for a large part of physics and the whole of chemistry are known"

No!

More bad guys!

Problems:

Hydrogen atom – small discrepancies

Relativity

Decay e.g. $n \rightarrow pe\nu_e$

How does a particle spontaneously evolve into several other particles?

➔ Quantum Field Theory

What do operators do

Operators operate on the wave function and spit out an answer.

e.g. $\hat{p}\psi$=(answer)ψ (for the experts - ψ should be an eigenfuction of $\hat{p}$

remember $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$

Lets take a plane wave (a typical wave function) $\psi(x) = Ae^{i(kx-\omega t)}$ where k and ω are numbers and x and t are variables
e.g. $\psi(x) = Ae^{i(4x-5t)}$

$$\hat{p}\psi(x)=\frac{\hbar}{i}\frac{d}{dx}Ae^{i(kx-\omega t)} = \hbar k\ \psi(x)$$

$$\hat{H}=\frac{\hat{p}^2}{2m} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}$$ (note we let V(2)=0) $$\hat{H}\psi(x)=\frac{\hbar^2 k^2}{2m}\ \frac{\hbar^2 k^2}{2m}$$ $$E_k = \frac{\hbar^2 k^2}{2m}$$ makes sense

How we do label $\psi(x)$? how about $\psi_k(x)$

Now going to QFT

$\psi_k(x) \rightarrow |k\rangle$ has energy $E_k = \frac{\hbar^2 k^2}{2m}$

Now for the SHO, or the Hydrogen atom. can we have any energy E_k? NO!

only certain energies allowed, so lets label these not with a k, but with the energy quantum number n

$|k\rangle \rightarrow |n\rangle$

Note - operators don't do anything to constants (e.g. $\frac{d}{dx}$)

Note- although I have said that $\psi_n(x) \rightarrow |n\rangle$, really its $\psi_n(x) \rightarrow \langle x|n\rangle$. $|n\rangle$ is a thing which lives in Hilbert space, we need the $\langle x|$ to put it into the "position" representation. As you might guess $\langle p|n\rangle$ is in the "momentum representaion". Its a lot clearer than writing t $\psi_n(x)$ and $\phi_n(p)$.

dirac notation - toward QFT (quantum field theory)

- E.g. look at energy levels of Hydrogen

  E_1, E_2, E_3 with wave functions $\psi_1(x)$, $\psi_2(x)$, $\psi_3(x)$,… $\psi_n(x)$

  - Remember we can specify x or p but not both – Can we write these as a function of p instead of x?? YES! $\phi_1(p)$, $\phi_2(p)$, $\phi_3(p)$,… $\phi_n(p)$
  - So generalize – call it $|n\rangle$ and specify $\psi_n(x)=\langle x|n\rangle$ and $\phi_n(p)=\langle p|n\rangle$
  - QFT: we now want $\boldsymbol{\psi}_n$ to be an operator so $\langle x|n\rangle=\langle 0|\,\boldsymbol{\psi}_n(x)|0\rangle$

hilbert space

- Now $|n\rangle$ is the "real thing"

  - X probability distribution = $|\langle x|n\rangle|^2$
  - p probability distribution = $|\langle p|n\rangle|^2$

now ψ and operator
QFT

energy (or number of particles) eigenfunction

position or momentum eigenfunctions

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Can we create or destroy particles? (energy quanta)

To Illustrate, let's use a Simple Harmonic Oscillator

$$V(x) = \frac{1}{2}kx^2 \qquad \omega = \sqrt{\frac{k}{m}} \qquad \boldsymbol{H} = \frac{\boldsymbol{p}^2}{2m} + \frac{m\omega^2 \boldsymbol{x}^2}{2} \qquad = (\boldsymbol{N} + \frac{1}{2})\hbar\omega$$

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 \psi + \frac{m\omega^2 x^2}{2}\psi$$

$$\boldsymbol{H}\left|n\right\rangle = (n + \frac{1}{2})\hbar\omega\left|n\right\rangle = \mathrm{E_n}\left|n\right\rangle$$

define $\boldsymbol{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{x} + \frac{i\boldsymbol{p}}{m\omega}\right) \qquad \boldsymbol{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\boldsymbol{x} - \frac{i\boldsymbol{p}}{m\omega}\right)$

† means hermitian conjugate

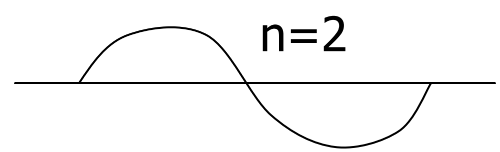
$$\boldsymbol{x} \rightarrow x \quad \boldsymbol{p} = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad \boldsymbol{N} \equiv \boldsymbol{a}^\dagger \boldsymbol{a}$$

$$\boldsymbol{a}^\dagger\left|n\right\rangle \sim \left|n+1\right\rangle \quad \boldsymbol{a}\left|n\right\rangle \sim \left|n-1\right\rangle \qquad \boldsymbol{N}\left|n\right\rangle = n\left|n\right\rangle$$

$\left|n\right\rangle$ is a state with E=$\mathrm{E_n} = (n + \frac{1}{2})\hbar\omega$

|n> is state with E_2

e.g. <x|2> is



SHO – raising and lowering operators (cont)

The reverse eqns for reference are $\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}^\dagger+\hat{a})$ and $\hat{p}=i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a}^\dagger-\hat{a})$

Then remembering that $[\hat{x},\hat{p}]=i\hbar$ $\quad [\hat{a},\hat{a}^\dagger]=\frac{i}{2\hbar}(-[\hat{x},\hat{p}]+[\hat{p},\hat{x}])=1$

more detail $[\hat{a},\hat{a}^\dagger]=\frac{m\omega}{2\hbar}[(\hat{x}+\frac{i\hat{p}}{m\omega})(\hat{x}-\frac{i\hat{p}}{m\omega})-(\hat{x}-\frac{i\hat{p}}{m\omega})(\hat{x}+\frac{i\hat{p}}{m\omega})]=$

$m\omega/(2\hbar)[(\hat{x}^2+\frac{\hat{p}^2}{m^2\omega^2}-\frac{i}{m\omega}\hat{x}\hat{p}+\frac{i}{m\omega}\hat{p}\hat{x})-(\hat{x}^2+\frac{\hat{p}^2}{m^2\omega^2}+\frac{i}{m\omega}\hat{x}\hat{p}-\frac{i}{m\omega}\hat{p}\hat{x})]=\frac{i}{2\hbar}(-[\hat{x},\hat{p}]+[\hat{p},\hat{x}])=1$

We can also define the number operator $\hat{N}=\hat{a}^\dagger\hat{a}=\frac{m\omega}{2\hbar}\left(\hat{x}^2+\frac{\hat{p}^2}{m^2\omega^2}+\frac{i}{m\omega}[\hat{x},\hat{p}]\right)=\frac{m\omega}{2\hbar}\hat{x}^2+\frac{\hat{p}^2}{2\hbar m\omega}-\frac{1}{2}=\frac{\hat{H}}{\hbar\omega}-\frac{1}{2}$

So $\hat{H}=\hbar\omega\left(\hat{N}+\frac{1}{2}\right)$

{ aside-We see immediately that $[\hat{H},\hat{N}]=0$ so we can have eigenkets of $\hat{H}$ which are also eigenkets of $\hat{N}$ and we will label the eigenkets of both $\hat{H}$ and $\hat{N}$ as $|n\rangle$}

We can define $|n\rangle$ so that

$\hat{N}|n\rangle=n|n\rangle$ so $\hat{N}|n\rangle=\left(\frac{\hat{H}}{\hbar\omega}-\frac{1}{2}\right)|n\rangle=n|n\rangle$ and it follows that $\hat{H}|n\rangle=\hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$ so the eigenvalues of $\hat{H}$ are $E_n=\hbar\omega\left(n+\frac{1}{2}\right)$

Now why do we call these thing raising and lowering operators? Lets see:

$[\hat{N},\hat{a}]=[\hat{a}^\dagger\hat{a},\hat{a}]=\hat{a}^\dagger[\hat{a},\hat{a}]+[\hat{a}^\dagger,\hat{a}]\hat{a}=-\hat{a}$ and $[\hat{N},\hat{a}^\dagger]=[\hat{a}^\dagger\hat{a},\hat{a}^\dagger]=\hat{a}^\dagger[\hat{a},\hat{a}^\dagger]+[\hat{a}^\dagger,\hat{a}^\dagger]\hat{a}=\hat{a}^\dagger$

different way

$[\hat{a},\hat{a}^\dagger]=1$ $\quad \hat{a}\hat{a}^\dagger=1+\hat{a}^\dagger\hat{a}$ $\quad [\hat{N},\hat{a}]=[\hat{a}^\dagger\hat{a},\hat{a}]=\hat{a}^\dagger\hat{a}\hat{a}-\hat{a}\hat{a}^\dagger\hat{a}=\hat{a}^\dagger\hat{a}\hat{a}-1\hat{a}-\hat{a}^\dagger\hat{a}\hat{a}=-\hat{a}$

rewite these $\hat{N}\hat{a}=-\hat{a}+\hat{a}\hat{N}$ $\quad \hat{N}\hat{a}^\dagger==\hat{a}^\dagger+\hat{a}^\dagger\hat{N}$

So $\hat{N}\hat{a}^\dagger|n\rangle=(\hat{a}^\dagger+\hat{a}^\dagger\hat{N})|n\rangle=\hat{a}^\dagger(n+1)|n\rangle=(n+1)\hat{a}^\dagger|n\rangle$ so $\hat{a}^\dagger|n\rangle\sim|n+1\rangle$ i.e. it raises it a step! There is a ~ there because there is still a normalization constant to figure out. Similarly

$\hat{N}\hat{a}|n\rangle=(-\hat{a}+\hat{a}\hat{N})|n\rangle=\hat{a}(n-1)|n\rangle=(n-1)\hat{a}|n\rangle$ so $\hat{a}|n\rangle\sim|n-1\rangle$ i.e. its a lowering operator.

SHO – raising and lowering operators (cont)

- Sometimes these things are call annihilation and creation operators because they either create of destroy one quantum of energy. Now lets figure out the normalization constant.

let c be the constant so $\hat{a}|n\rangle = c|n-1\rangle$. We want both $\langle n|n\rangle=1$ and $\langle n-1|n-1\rangle=1$ so

$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = |c|^2\langle n-1|n-1\rangle = |c|^2$ but $\langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n|\hat{N}|n\rangle = n\langle n|n\rangle = n$ so $c=\sqrt{n}$

Also we know $[\hat{a},\hat{a}^\dagger]=1 \rightarrow \hat{a}\hat{a}^\dagger = 1+\hat{a}^\dagger\hat{a} = 1+\hat{N}$

let $\hat{a}^\dagger|n\rangle = c|n+1\rangle$ so $\langle n|\hat{a}\hat{a}^\dagger|n\rangle = |c|^2\langle n+1|n+1\rangle = |c|^2$ but $\langle n|\hat{a}\hat{a}^\dagger|n\rangle = \langle n|\hat{N}+1|n\rangle = (n+1)\langle n|n\rangle = n+1$ so $c=\sqrt{n+1}$ and finally summing up

$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

Now if we can just figure out some lowest energy state, we can just bootstrap our way up by using the raising operators!

So lets see if we can formulate an argument.

First we can show that n must be an integer: For the moment lets call $\hat{a}|n\rangle = |\alpha\rangle$. Now we dont know if this thing is normalized (it isn't as we know from above) but we do know that $\langle\alpha|\alpha\rangle \geq 0$ $\rightarrow$ $n = \langle n|\hat{N}|n\rangle = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle\alpha|\alpha\rangle \geq 0$ so $n \geq 0$

Next we know that if we start is some $|n\rangle$ then we can use the lowering operator as follows

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

SHO - eigenfunctions

$\hat{a}^2|n\rangle=\sqrt{n(n-1)}\,|n\text{-}2\rangle$

$\hat{a}^3|n\rangle=\sqrt{n(n-1)(n-2)}\,|n\text{-}3\rangle$ etc. This can keep going forever, UNLESS at some point the number inside the ket is zero, then the series will terminate.

Putting this together with the fact that $n\geq 0$ means that the series MUST terminate at n=0. Therefore n=0 is the lowest energy state where we want to start our bootstrap.

So the ground state is $|0\rangle$ with energy eigenvalue $E_0 = \frac{1}{2}\hbar\omega$ It is striking that the lowest energy state is NOT zero! This

has implications as we ask how what things look like at zero temperature - what the state of the vacuum is, dark energy, the casmir force etc etc.

Now we can just work our way up to all the other eigenstataes:

$|1\rangle=\hat{a}^\dagger\,|0\rangle \quad E_1 = \frac{3}{2}\hbar\omega$

$|2\rangle=\frac{\hat{a}^\dagger}{\sqrt{2}}\,|1\rangle = \frac{(\hat{a}^\dagger)^2}{\sqrt{2!}}\,|0\rangle \quad E_2 = \frac{5}{2}\hbar\omega$

$|3\rangle=\frac{\hat{a}^\dagger}{\sqrt{3}}\,|2\rangle = \frac{(\hat{a}^\dagger)^3}{\sqrt{3!}}\,|0\rangle \quad E_2 = \frac{7}{2}\hbar\omega$

$|n\rangle= \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle \quad E_n = \left(n+\frac{1}{2}\right)\hbar\omega$

tation

We avoided solving the differential eqn which would give us the wave function in position representation - because it is a pain.

We can go ahead and solve the diff - eqn by force. It turns out, this raising and lowering operator also makes finding the wave functions easier

Lets first write $\hat{a}^\dagger$ and $\hat{a}$ out in the "position representation"

$$\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right)=\sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{\hbar}{m\omega}\frac{d}{dx}\right) \quad \text{and} \quad \hat{a}^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i\hat{p}}{m\omega}\right)=\sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{\hbar}{m\omega}\frac{d}{dx}\right)$$

Remember we decided that n=0 was the ground state and we know $\hat{a}\,|0\rangle=\sqrt{0}\,|-1\rangle=0$. Writing this in position representation gives

$$\langle x'\,|\,\hat{a}\,|\,0\rangle=\sqrt{\frac{m\omega}{2\hbar}}\left\langle x'\left|\hat{x}+\frac{i\hat{p}}{m\omega}\right|0\right\rangle=\sqrt{\frac{m\omega}{2\hbar}}\left(\langle x'\,|\,\hat{x}\,|\,0\rangle+\frac{i}{m\omega}\left\langle x'\,|\,\hat{p}\,\middle|\,0\right\rangle\right)$$

Now $\langle x'\,|\,\hat{x}\,|\,0\rangle=\langle x'\,|\,\hat{x}\,|\,0\rangle^{\dagger\dagger}=\langle 0\,|\,\hat{x}^\dagger\,|\,x'\rangle^{\dagger}=\langle 0\,|\,\hat{x}\,|\,x'\rangle^{\dagger}=x'\langle 0\,|\,x'\rangle^{\dagger}=x'\langle x'|0\rangle$ and $\left\langle x'\,|\,\hat{p}\,\middle|\,0\right\rangle=\frac{\hbar}{i}\frac{d}{dx'}\langle x'\,|\,0\rangle$ so

$$\langle x'\,|\,\hat{a}\,|\,0\rangle=\sqrt{\frac{m\omega}{2\hbar}}\left(x'\langle x'|0\rangle+\frac{\hbar}{m\omega}\frac{d}{dx'}\langle x'|0\rangle\right)$$

and finally $\left(x'+x_0^2\frac{d}{dx'}\right)\langle x'|0\rangle=0$ where $x_0=\sqrt{\frac{\hbar}{m\omega}}$

The solution to this is a gaussian:

$\langle x'\,|\,0\rangle=\left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right)\exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right]$ (check it). We can just use the raising operator to find the rest:

$$\langle x'|1\rangle=\langle x'\,|\,\hat{a}^\dagger\,|\,0\rangle=\sqrt{\frac{m\omega}{2\hbar}}\left\langle x'\left|\hat{x}-\frac{i\hat{p}}{m\omega}\right|0\right\rangle=\sqrt{\frac{m\omega}{2\hbar}}\left(\langle x'\,|\,\hat{x}\,|\,0\rangle-\frac{i}{m\omega}\left\langle x'\,|\,\hat{p}\,\middle|\,0\right\rangle\right)=\sqrt{\frac{m\omega}{2\hbar}}\left(x'\langle x'|0\rangle-\frac{\hbar}{m\omega}\frac{d}{dx'}\langle x'|0\rangle\right)$$

$$=\left(\frac{1}{\sqrt{2}\,x_0}\right)\left(x'-x_0^2\frac{d}{dx'}\right)\langle x'|0\rangle$$

We can going and in general $\langle x'\,|\,n\rangle=\left(\frac{1}{\pi^{1/4}\sqrt{2^n\,n!}}\right)\left(\frac{1}{x_0^{n+1/2}}\right)\left(x'-x_0^2\frac{d}{dx'}\right)^n\exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right]$

$\langle x'\,|\,n\rangle=A_n\,\mathcal{H}_n(\xi)\,e^{-\xi^2}$ where $\xi=\frac{m\omega_0}{\hbar}x^2$, $A_n=\left(2^n\,n!\,\sqrt{\pi}\right)^{-1/2}$ and $\mathcal{H}_n(\xi)$ are the Hermite polynomials

so $\langle x'\,|0\rangle=A_0\,e^{-\frac{\xi^2}{2}}$ $\langle x'\,|1\rangle=A_1\,2\,\xi\,e^{-\frac{\xi^2}{2}}$ $\langle x'\,|2\rangle=A_2\left(4\,\xi^2-2\right)e^{-\frac{\xi^2}{2}}$ etc

Lets figure out some stuff for the ground state $|0\rangle$

$\langle\hat{x}\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle\hat{a}^\dagger + \hat{a}\rangle = 0$ and $\langle\hat{p}\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,\langle\hat{a}^\dagger - \hat{a}\rangle = 0$

check heisenberg

$$\langle\hat{x}^2\rangle = \frac{\hbar}{2m\omega}\langle\hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\rangle = \frac{\hbar}{2m\omega}\left\langle\hat{a}\,\hat{a}^\dagger\right\rangle = \frac{\hbar}{2m\omega}\langle 0|\,\hat{a}\,\hat{a}^\dagger\,|0\rangle = \frac{\hbar}{2m\omega}(0+1)\left\langle 1|\,1\right\rangle = \frac{\hbar}{2m\omega} = \frac{x_0^2}{2}$$

$$\langle\hat{p}^2\rangle = -\frac{m\hbar\omega}{2}\langle\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\rangle = \frac{m\hbar\omega}{2}\left\langle\hat{a}\,\hat{a}^\dagger\right\rangle = \frac{m\hbar\omega}{2}\langle 0|\,\hat{a}\,\hat{a}^\dagger\,|0\rangle = \frac{m\hbar\omega}{2}(0+1)\left\langle 1|\,1\right\rangle = \frac{m\hbar\omega}{2}$$

The kinetic energy $\left\langle\frac{\hat{p}^2}{2m}\right\rangle = \frac{\hbar\omega}{4} = \frac{\langle\hat{H}\rangle}{2}$ and $\left\langle\frac{m\omega^2\hat{x}^2}{2}\right\rangle = \frac{m\omega^2}{2}\frac{\hbar}{2m\omega} = \frac{\hbar\omega}{4} = \frac{\langle\hat{H}\rangle}{2}$ so the energy is split for the ground state between the potential energy and kinetic energy as expected from the virial theorem

$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$ $\quad \Delta x = \sqrt{\frac{m\hbar\omega}{2}}$ $\quad \Delta x\,\Delta p = \frac{\hbar}{2}$ (i.e. it is at the minimum uncertainty by Heisenberg)

What to these wave functions look like?.

graph some

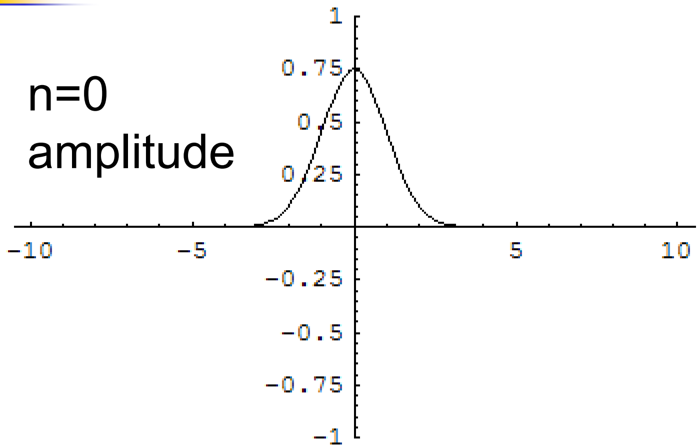
Here are a few

$$\langle x'|0\rangle = \frac{e^{-\frac{x^2}{2}}}{\sqrt[4]{\pi}} \quad \langle x'|1\rangle = \frac{\sqrt{2}\,e^{-\frac{x^2}{2}}\,x}{\sqrt[4]{\pi}} \quad \langle x'|2\rangle = \frac{e^{-\frac{x^2}{2}}(2x^2-1)}{\sqrt{2}\,\sqrt[4]{\pi}} \quad \langle x'|3\rangle = \frac{e^{-\frac{x^2}{2}}\,x(2x^2-3)}{\sqrt{3}\,\sqrt[4]{\pi}} \quad \langle x'|4\rangle = \frac{e^{-\frac{x^2}{2}}(4(x^2-3)x^2+3)}{2\sqrt{6}\,\sqrt[4]{\pi}}$$

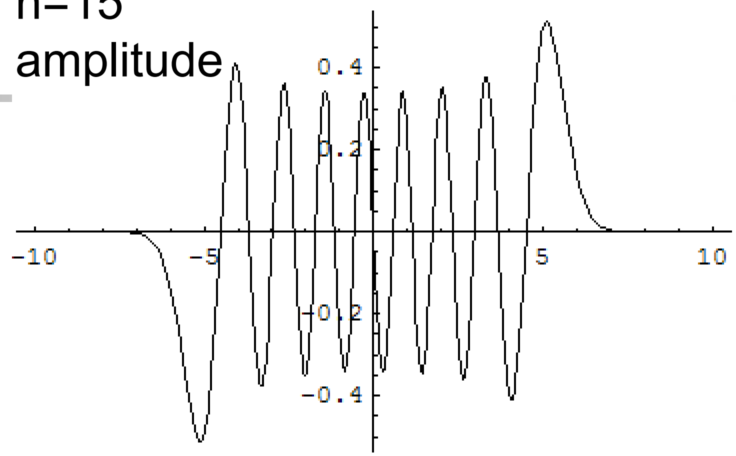
I have plotted $\langle x'|n=15\rangle$. (See qm19work for the mathematica to figure out the wave functions and plot it)

$$\langle x'|15\rangle = \frac{e^{-\frac{x^2}{2}}\,x(2x^2(2(2x^2(2(8x^6-420x^4+8190x^2-75075)x^2+675675)-2837835)x^2+4729725)-2027025)}{30240\sqrt{715}\,\sqrt[4]{\pi}}$$

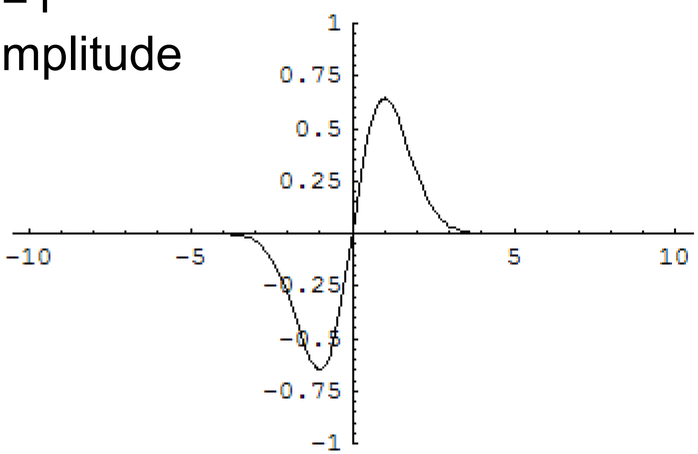
n=0
amplitude

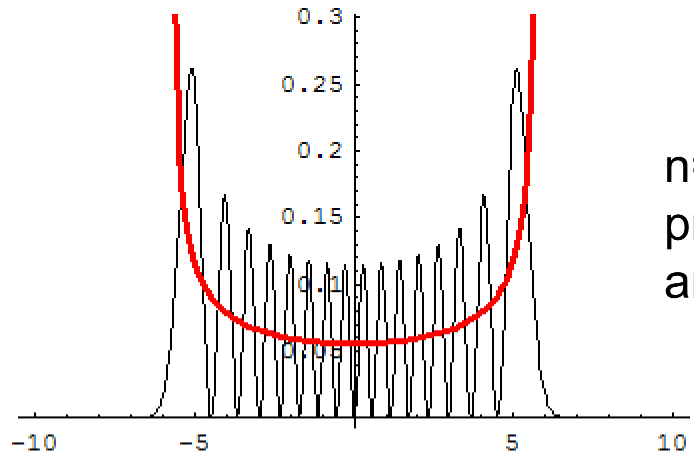


n=15
amplitude

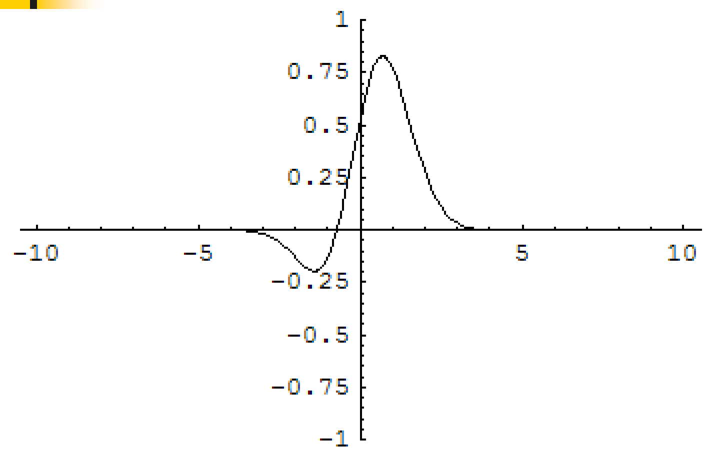


n=1
amplitude

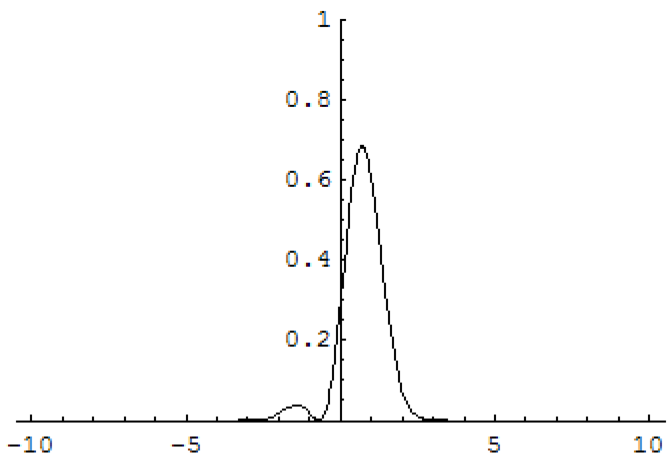




n=15
prob ~
amp^2



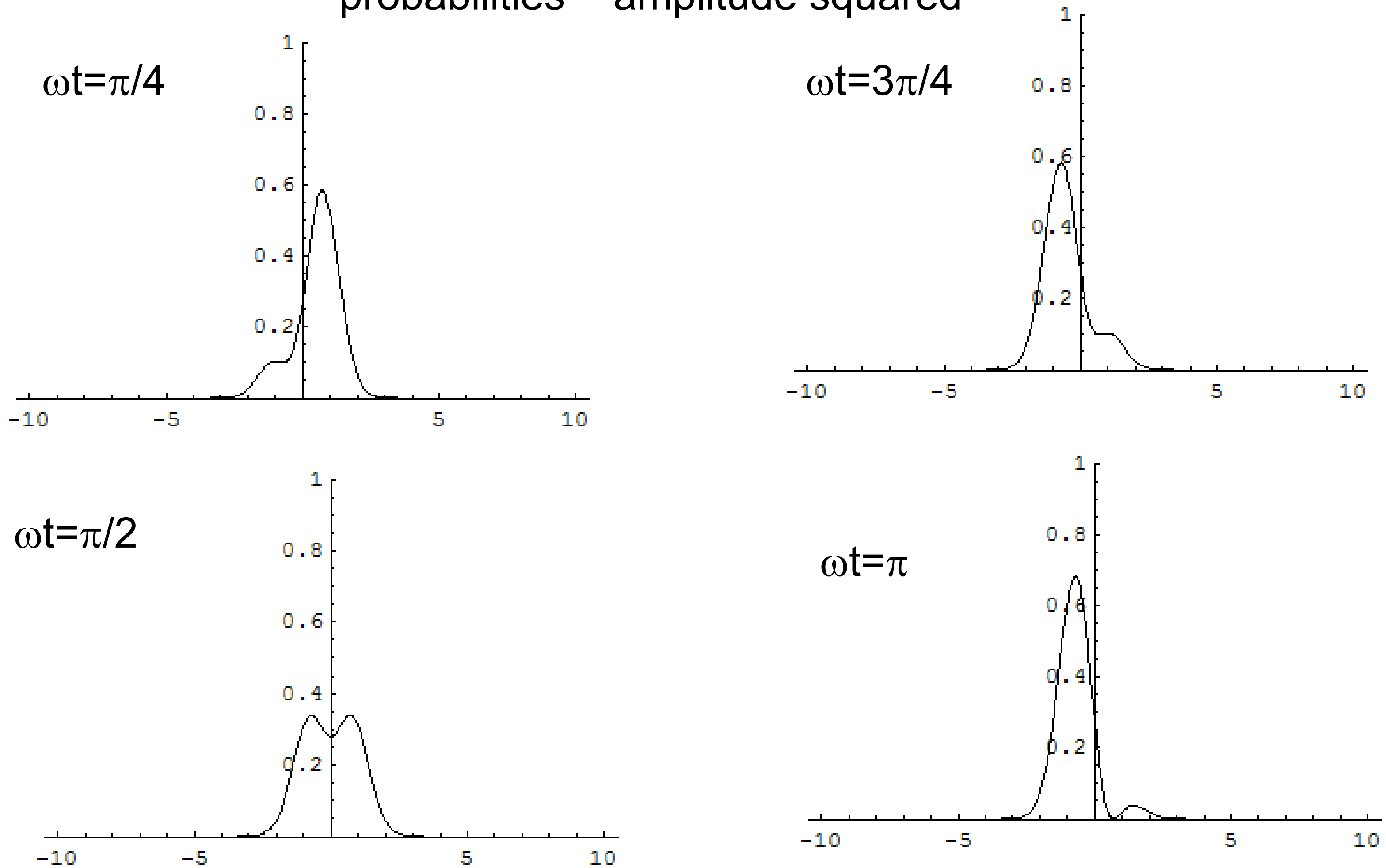
n=0 + n=1
amplitude



n=0 + n=1
prob

time dependence for fun

probabilities ~ amplitude squared



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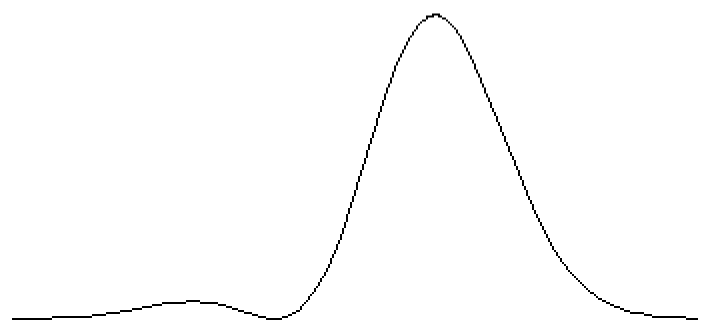
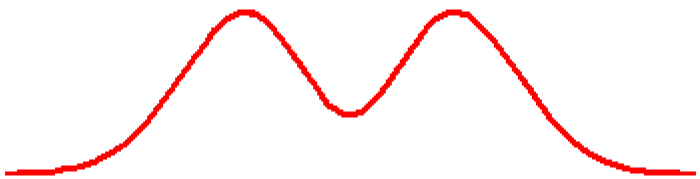
position ket



momentum ket



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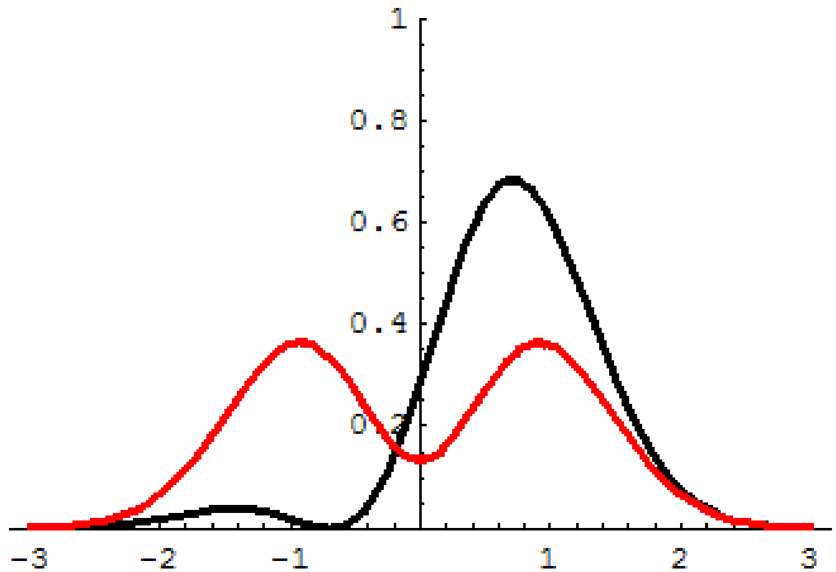
both



Now skip lots of steps-creation and annihilation operators

Let $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$ be the basic "things" and $|0\rangle$ be the vacuum

Then $\boldsymbol{a}^\dagger|0\rangle \sim |1\rangle$ i.e. a is an operator which creates 1 quanta

Later we will write $\boldsymbol{a}^\dagger \to \phi^\dagger$ which we call a field operator

An aside: Really the typical way we do this is as follows:

Start from E and M, with the classical radiation field

$\Rightarrow$ Quantize $\Rightarrow$ Field operators

E and M $\quad Energy = \int \left(B^2 + E^2\right) d^3r \quad$ Remember??

Define $\mathbf{P}\left(B,E\right)$ and $\mathbf{X}(B,E)$

really $B = \nabla \times A$ and $E = -\frac{1}{c}\frac{\partial A}{\partial t}$

k=momentum of γ

α=1,0,-1 polarization of photon

$$\boldsymbol{A} = \sum_k \sum_\alpha \left[a_{k,\alpha}\varepsilon^\alpha e^{i(kx-\omega t)} + a^\dagger_{k,\alpha}\varepsilon^\alpha e^{-i(kx-\omega t)} \right]$$

$$\mathbf{P}_{k,\alpha} = -\tfrac{i\omega}{c}\left(\boldsymbol{a}_{k,\alpha} - \boldsymbol{a}^\dagger_{k,\alpha}\right) \quad \mathbf{X}_{k,\alpha} = \tfrac{1}{c}\left(\boldsymbol{a}_{k,\alpha} + \boldsymbol{a}^\dagger_{k,\alpha}\right)$$

Particles and the Vacuum

Apply to photons

$$\left[\mathbf{X}_{k\alpha},\mathbf{P}_{k\alpha}\right]=i\hbar \quad \Rightarrow \; \boldsymbol{a}^{\dagger}\left|n\right\rangle \sim \left|n+1\right\rangle \;\; \boldsymbol{a}\left|n\right\rangle \sim \left|n-1\right\rangle$$

$\boldsymbol{N}=\boldsymbol{a}^{\dagger}\boldsymbol{a} \quad \boldsymbol{N}\left|n\right\rangle = n\left|n\right\rangle$ number operator

$$\boldsymbol{H}=\sum_{k}\sum_{\alpha}\left(\boldsymbol{N}_{k,\alpha}+\frac{1}{2}\right)\hbar\omega \qquad \omega = ck$$

$$Total\_Energy=\sum_{k}\sum_{\alpha}\left(number\_of\_photons+\frac{1}{2}\right)Energy\_of\_photon$$

Sakurai: " The quantum mechanical excitation of the radiation field can be regarded as a particle, the photon, with mass 0, and spin 1.

Field not excited $\Rightarrow$ vacuum $\Rightarrow$ $\left|0\right\rangle$

Wiggle (excite) the field $\Rightarrow$ particle(s) $\Rightarrow$ $\left|n\right\rangle$

Vacuum has energy???

$$\boldsymbol{H} = \sum_k \sum_\alpha \left(\boldsymbol{N}_\gamma + \frac{1}{2} \right) \hbar\omega$$

vacuum when $N_\gamma = 0$

$$E_{vacuum} = \sum_k \sum_\alpha \left(\frac{1}{2} \right) \hbar\omega \quad ???$$

- The Vacuum has energy?
  - Experiment – measure Force between two plates in a vacuum F=dEnergy(vacuum)/dx
  - Done is last several years – agrees with prediction!
  - Dark energy? – right idea but quantitatively a BIG discrepancy

What tools to we have to build a theory?

- To Build theories (an effective Lagrangian) we invoke fundamental symmetries.
  - Why? -
    - because it works
    - and it seems right somehow??? “ H. Georgi”
  - Examples
    - translational invariance ✹momentum conservation
    - rotational invariance ✹ angular momentum conservation
    - Local gauge invariance (phase change) ✹ EM forces
- In the standard theories we essentially always start with a massless theory –
  - Mass comes about from a breaking of symmetry giving rise to a complicated vacuum.

Lagrangian Formulation

- Compact, Formal way to get eqns. of motion (F=ma)
  - Lagrangian L=T-V=Potential E – Kinetic E (Hamiltonian E=H=T+V)
  - Lagrange's eqn – just comes from some math

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right)-\frac{dL}{dq}=0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{dL}{dv}\right)-\frac{dL}{dx}=0$$

$e.g.$ Simple Harmonic Oscillator (A spring)

V=Pot. Energy = $\frac{1}{2}kx^2$

$$L=\frac{1}{2}mv^2-\frac{1}{2}kx^2 \qquad \frac{dL}{dv}=mv \qquad \frac{dL}{dx}=-kx$$

$$\frac{d}{dt}(mv)+kx=0 \quad ma=-kx \quad F=ma!!!$$

Symmetries - Example

- L is independent of x (translational invariance)
  - I.e. physics doesn't depend on position

If we choose $L = \frac{1}{2}mv^2 \quad \Rightarrow$ it is independent of position

then from Lagranges eqn. $\frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right) - \frac{dL}{dq} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{dL}{dv}\right) - \frac{dL}{dx} = 0$

$\frac{dL}{dx} = 0 \quad \frac{dL}{dv} = mv \qquad \frac{d}{dt}(mv) = 0 \quad \Rightarrow \quad \frac{dp}{dt} = 0$

i.e. p is constant !, momentum is conserved!

Symmetries $\Leftrightarrow$ Conservation Laws

space invariance $\rightarrow$ conservation of p

time invariance $\rightarrow$ conservation of E

rotational invariance $\rightarrow$ conservation of angular momentum

More on notation

$\hat{H}$ or $\hat{p}$ or $\hat{a}^\dagger$ are operators – they do stuff

ket's $|\phi\rangle$ are states – they live in hilbert space, same thing with bra's $\langle\psi|$

only bra – kets $\langle\psi|\hat{H}|\phi\rangle$ or $\langle\psi|\phi\rangle$ are physical

So when we want to think about what we can

measure we will always look at stuff like $\langle\psi|\phi\rangle$ or $\langle\psi|\hat{H}|\phi\rangle$

Note – $\langle\psi|\psi\rangle = \langle\psi|x\rangle\langle x|\psi\rangle = |\langle x|\psi\rangle|^2 = \psi(x)|^2$

Note – when and operator operates on a ket, it makes a new ket e.g.

$$|\phi\rangle = \hat{a}|\phi\rangle$$

Note- what does it mean when there are several operators in a row?

$$\langle\psi|\hat{a}^\dagger\hat{H}\hat{a}|\phi\rangle = \langle\psi'|\hat{H}|\phi'\rangle = \langle\psi|\hat{U}^\dagger\hat{H}\hat{U}|\phi\rangle$$

We can consider a operator operating forward as it conjugate operating backwards e.g.

$|\phi\rangle = \hat{a}^\dagger|\phi\rangle$ then

$$\langle\phi| = \langle\phi|\hat{a}^\dagger$$

Note: read $\langle\hat{H}\rangle$ as "average value of $\hat{H}$" (of couse the problem is average over what? - that has to be understood from its context...)

Note on Symmetries

now $\hat{H}^\dagger = \hat{H}$

Let $\hat{U}$ be some transformation (i.e.translation in space, angle, time, ...)

Now $\hat{H}$ is invariant under rotations

$$\langle \psi |\hat{H}| \phi \rangle = \langle \psi' |\hat{H}| \phi' \rangle = \langle \psi |\hat{U}^\dagger \hat{H} \hat{U}| \phi \rangle$$

So $\quad \hat{H} = \hat{U}^\dagger \hat{H} \hat{U} \quad$ and finally $\left[\hat{H}, \hat{U}\right] = 0$

Time dependece of Expecation Values

$$\frac{d\langle \hat{U} \rangle}{\mathrm{dt}} = \frac{d\langle \psi, t \,|\hat{U}|\, \psi, t \rangle}{\mathrm{dt}}$$

$$|\,\psi, t\rangle \;\rightarrow\; |\,\psi', t\rangle = \hat{U}\,|\,\psi, t\rangle \quad \text{and} \quad \langle \psi', t\,| = \langle \psi, t\,|\,\hat{U}^\dagger$$

$$|\langle \psi | \phi \rangle|^2 = |\langle \psi' | \phi' \rangle|^2 = |\langle \psi |\hat{U}^\dagger \hat{U}| \phi \rangle|^2$$

Assume that $\hat{U}$ does not have an explicite time dependence (e.g. it could me momentum, angular momentum-which we will learn about later, spin...)

we can write the time dependent Schroedinger eqn.

$$i\hbar \frac{d}{dt}|\psi, t\rangle = \hat{H}|\psi, t\rangle$$ and the hermitian conjugate of this eqn

$$-i\hbar \frac{d}{dt}\langle \psi, t| = \langle \psi, t|\hat{H}$$ (remember that $\hat{H}$ is hermitian)

$$\frac{d\langle \psi,t|\hat{U}|\psi,t\rangle}{dt} = \frac{d\langle\psi,t|}{dt}\hat{U}|\psi,t\rangle + \langle\psi,t|\hat{U}\frac{d|\psi,t\rangle}{dt} =$$

$$\frac{i}{\hbar}\langle\psi,t|\hat{H}\hat{U}|\psi,t\rangle - \frac{i}{\hbar}\langle\psi,t|\hat{U}\hat{H}|\psi,t\rangle = \frac{i}{\hbar}\langle\psi,t|\left[\hat{H},\hat{U}\right]|\psi,t\rangle$$

So finally we have

$$\frac{d\langle\hat{U}\rangle}{dt} = \frac{i}{\hbar}\left\langle\left[\hat{H},\hat{U}\right]\right\rangle$$ so

if $\left[\hat{H},\hat{U}\right]=0$ then $\frac{d\langle\hat{U}\rangle}{dt}=0$

This is an important theorem. It means that if any observable $\hat{U}$ commutes with $\hat{H}$ then its expectation value does not change with time - that is - it is a constant of motion.

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Symmetries and Conservation laws

This then leads to the connection between the properties of space and conservation laws. As I will show you, the homogeneity of space will lead to momentum conservation - i.e physics doesnt change depending on where you are, simillarly the fact that physics doesn't change when you rotate things leads to angular momentum conservation, and the fact that physics doesnt change with time will lead to energy conservation. There are other symmetry principles which lead to other conservation laws as well. Here I will show you some.

Note: remember the Taylor expansion

$$f(x)=f(a)+(x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)+\ldots$$

Lets think about translation. We will deal with infinitesimal translations for ease, but we can always generalize to finite translations. Lets define a translation operator

$\hat{\mathcal{T}}(\delta x)|x\rangle=|x+\delta x\rangle$ (the book uses $\hat{D}$)

To first order $\hat{\mathcal{T}}(\delta x) =\hat{1}-i\delta x\frac{\hat{p}_x}{\hbar}$ since $\hat{\boldsymbol{P}}_x= \frac{\hbar}{i}\frac{d}{dx}$

Now lets take some ket $|\psi\rangle$. What does it mean that physics is the same everywhere - that is, that it is translationaly invariant? Well it means that if we get some answer for one of our invariant observables (lets take $\hat{H}$) then if we figure out that observable somwhere else, it should give the same answer. So for example, lets assume that $\hat{H}$ is the free particle Hamiltonian which is just $\hat{p}^2/2m$. It has no dependence on position. This means that if we take a ket $|\psi\rangle$ and calculate $\langle\psi|\hat{H}|\psi\rangle$ and then move the ket somewhere else using the translation operator, it should give the same answer.

So lets first look at $\langle\hat{H}\rangle$ and the original location $\langle\psi\,|\,\hat{H}\,|\,\psi\rangle$. Now if space is homogeneous the this should be the same as the expectation value for $\hat{T}\,|\psi\rangle$, that is $\left\langle\psi\middle|\hat{T}^{\dagger}\hat{H}\hat{T}\middle|\psi\right\rangle$. So if $\left\langle\psi\middle|\hat{T}^{\dagger}\hat{H}\hat{T}\middle|\psi\right\rangle=\langle\psi\,|\,\hat{H}\,|\,\psi\rangle$, this means that

$\hat{T}^{\dagger}\hat{H}\hat{T}=\hat{H}$, so $\hat{H}\hat{T}=\hat{T}\hat{H}$ $\rightarrow[\hat{H},\hat{T}]=0\rightarrow[\hat{H},\hat{p}_x]=0$

But using the theorem above we get that $\frac{d\langle\hat{p}_x\rangle}{dt}=[\hat{H},\hat{p}_x]=0$ and momentum is conserved.

We can do the same where we replace $\hat{T}$ with the unitary time evolution operator $\hat{H}=i\hbar\frac{d}{dt}$ from the Schroedinger eqn and get $\frac{d\langle E\rangle}{dt}=0$, the conservation of energy. Later we will have a rotation operator which will lead to the conservation of angular momentum.

Aside – for electrons

Add relativity

$$E^2 = p^2c^2 + m^2c^4$$

units c=1, $\hbar$=1

$$E^2 = p^2 + m^2$$

guess mass like potential energy

guess $H = Kinetic\_energy + mass\_term$

$$L = Kinetic\_energy - mass\_term$$

Guess $\mathrm{L}_{electron} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi$

Probability = $\bar{\psi}\psi \sim mass$

- So we now build theories which have symmetries

Require other symmetries
Local gauge invariance ⇒ Forces!

$$L_{electron} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \qquad \partial\!\!\!/ = \gamma^{\mu}\partial_{\mu}$$

Probability = $\bar{\psi}\psi$

what about the phase of $\psi(x)$??

Ans: Its arbitary $\psi' = e^{i\Lambda}\psi$, Λ=constant

$L_{electron} = L'_{electron}$ $so\ L_{electron}$ is invariant under (global) phase (guage) transfromation

What about if we made Λ dependent on the position ??? $\Lambda=\Lambda$(x)

$\psi' = e^{i\Lambda(x)}\psi$ local gauge transformation ⇒ check it

mass term: $m\bar{\psi}\psi \to m\bar{\psi}e^{-i\Lambda(x)}e^{i\Lambda(x)}\psi = m\bar{\psi}\psi$ so mass term is ok

1st term: $i\bar{\psi}\partial\!\!\!/\psi \to i\bar{\psi}e^{-i\Lambda(x)}\left[\partial\!\!\!/\left(e^{i\Lambda(x)}\psi(x)\right)\right]$

$$= i\bar{\psi}e^{-i\Lambda(x)}\left[e^{i\Lambda(x)}\partial\!\!\!/\psi(x) + \left(\partial\!\!\!/ e^{i\Lambda(x)}\right)\bar{\psi}\right]$$

So we have a problem....

Solution (a crazy one)

Add a new term to the Lagrangian

$$\mathfrak{L}_{New} = i\bar{\psi}\left[\partial\!\!\!/ + i\gamma^{\mu}A_{\mu}\right]\psi - m\bar{\psi}\psi$$

$$e.g.\ \ B = \nabla \times A$$

Local Gauge transformation :

$$\psi(x) \to e^{i\Lambda(x)}\psi(x)$$

$$A_{\mu} \to A_{\mu} - \partial_{\mu}\Lambda(x)$$

note: $\nabla \times A' = \nabla \times (A - \nabla\Lambda) = \nabla \times A = B$ so its OK.

now check $\mathfrak{L}_{New}$ 1st term:

$$i\bar{\psi}\left[\partial\!\!\!/ + i\gamma^{\mu}A_{\mu}\right]\psi \to i\bar{\psi}e^{-i\Lambda(x)}\left[\partial\!\!\!/ + i\gamma^{\mu}A_{\mu} - i\gamma^{\mu}\left(\partial_{\mu}\Lambda(x)\right)\right]\left(e^{i\Lambda(x)}\psi(x)\right) =$$

$$i\bar{\psi}e^{-i\Lambda(x)}\left[e^{i\Lambda(x)}\partial\!\!\!/\psi + e^{i\Lambda(x)}\left(\partial\!\!\!/\Lambda(x)\right)\psi + i\gamma^{\mu}A_{\mu}e^{i\Lambda(x)}\psi - i\left(\partial\!\!\!/\Lambda(x)\right)e^{i\Lambda(x)}\psi(x)\right]$$

$$= i\bar{\psi}\partial\!\!\!/\psi + i\bar{\psi}\gamma^{\mu}A_{\mu}\psi = i\bar{\psi}\left[\partial\!\!\!/ + i\gamma^{\mu}A_{\mu}\right]\psi$$

it works!

So What is this thing A?

A nice way to write out A is

$$\boldsymbol{A} = \sum_k \sum_\alpha \left[a_{k,\alpha} \varepsilon^\alpha e^{i(kx-\omega t)} + a^\dagger_{k,\alpha} \varepsilon^\alpha e^{-i(kx-\omega t)} \right]$$

$$\mathbf{P}_{k,\alpha} = -\tfrac{i\omega}{c}\left(\boldsymbol{a}_{\boldsymbol{k,\alpha}} - \boldsymbol{a}^\dagger_{\boldsymbol{k,\alpha}}\right) \quad \mathbf{X}_{k,\alpha} = \tfrac{1}{c}\left(\boldsymbol{a}_{\boldsymbol{k,\alpha}} + \boldsymbol{a}^\dagger_{\boldsymbol{k,\alpha}}\right)$$

if we work on it we will find $\left[\mathbf{X}_{k,\alpha}, \mathbf{P}_{k,\alpha}\right] = i\hbar$

and define

$B = \nabla \times A$ and $E = -\dfrac{1}{c}\dfrac{\partial A}{\partial t}$

Ans: Photons?

We have Electricity and Magnetism?

really?

What happens if we give mass to A?

$$\mathfrak{L}_{\gamma mass} = m_\gamma A^\mu A_\mu$$

Is this term Guage invariant? Try it!

remember $A_\mu \to A_\mu - \partial_\mu \Lambda\left(x\right)$

$$m_\gamma A^\mu A_\mu \to m\left(A^\mu - \partial^\mu \Lambda(x)\right)\left(A_\mu - \partial_\mu \Lambda(x)\right) \neq m_\gamma A^\mu A_\mu$$

- The term is NOT gauge invariant
  - So do we throw out gauge invariance??
  - NO – we set m_γ=0 ➔ photons are masses
- IS it really electricity ??? – photons? (looks like a duck…)
- Agrees with experiment! – you can figure out $F=q_1q_2/r^2$ etc – can you get what we know as QED? The feynman diagrams?

the QED lagrangian

Now you may wonder if the A field (photon) have a kinetic energy term. It does. It is

$-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$ (were the μ and ν are summed over 0-3) and

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

The total Lagrangian now is

$\text{L=T-V=T}_{\text{electron}} + \text{T}_{\text{photon}} \text{-Mass}_{\text{electron}} - \text{V}_{\text{interaction}}$

$$L_{QED} = i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} - m\bar{\psi}\psi + \bar{\psi}\gamma^\mu A_\mu \psi$$

where $\text{V}_{\text{interaction}} = \bar{\psi}\gamma^\mu A_\mu \psi = \bar{\psi} A\!\!\!/\psi$

Perturbation theory

- We begin with an unperturbed Hamiltonian *H0*, which is also assumed to have no time dependence. It has known energy levels and eigenstates, arising from the time-independent Schrödinger equation:

$$H_0|n^{(0)}\rangle = E_n^{(0)}|n^{(0)}\rangle \quad , \quad n = 1, 2, 3, \cdots$$

- For simplicity, we have assumed that the energies are discrete. The (0) superscripts denote that these quantities are associated with the unperturbed system.
- We now introduce a perturbation to the Hamiltonian. Let *V* be a Hamiltonian representing a weak physical disturbance, such as a potential energy produced by an external field. (Thus, *V* is formally a Hermitian operator.) Let λ be a dimensionless parameter that can take on values ranging continuously from 0 (no perturbation) to 1 (the full perturbation). The perturbed Hamiltonian is
  - $H = H0 + \lambda V$
- The energy levels and eigenstates of the perturbed Hamiltonian are again given by the Schrödinger equation:

$$\left(H_0 + \lambda V\right)|n\rangle = E_n|n\rangle.$$

- Our goal is to express *En* and |*n*> in terms of the energy levels and eigenstates of the old Hamiltonian. If the perturbation is sufficiently weak, we can write them as power series in *λ*:

$$|n\rangle = |n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \cdots$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots$$

- Plugging the power series into the Schrödinger equation, we obtain

$$(H_0 + \lambda V)\left(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \cdots\right)$$
$$= \left(E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots\right)\left(|n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \cdots\right)$$

We will set all the λ to 1. Its just a way to keep track of the order of the correction

- Expanding this equation and comparing coefficients of each power of *λ* results in an infinite series of simultaneous equations. The zeroth-order equation is simply the Schrödinger equation for the unperturbed system. The first-order equation is

$$H_0|n^{(1)}\rangle + V|n^{(0)}\rangle = E_n^{(0)}|n^{(1)}\rangle + E_n^{(1)}|n^{(0)}\rangle$$

- Multiply through by <*n(0)*|. The first term on the left-hand side cancels with the first term on the right-hand side. (Recall, the unperturbed Hamiltonian is hermitian). This leads to the first-order energy shift:

$$E_n^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle$$

- This is simply the expectation value of the perturbation Hamiltonian while the system is in the unperturbed state. This result can be interpreted in the following way: suppose the perturbation is applied, but we keep the system in the quantum state |*n(0)*>, which is a valid quantum state though no longer an energy eigenstate. The perturbation causes the average energy of this state to increase by <*n(0)*| *V*|*n(0)*>. However, the true energy shift is slightly different, because the perturbed eigenstate is not exactly the same as |*n(0)*>. These further shifts are given by the second and higher order corrections to the energy.

We get

$$E_n = E_n^{(0)} + \langle n^{(0)}|V|n^{(0)}\rangle + \sum_{k\neq n} \frac{|\langle k^{(0)}|V|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} + \cdots$$

$E_n^{(1)}$ $E_n^{(2)}$

$$E_n^{(3)} = \sum_{k\neq n}\sum_{m\neq n} \frac{\langle n^{(0)}|V|m^{(0)}\rangle\langle m^{(0)}|V|k^{(0)}\rangle\langle k^{(0)}|V|n^{(0)}\rangle}{\left(E_m^{(0)} - E_n^{(0)}\right)\left(E_k^{(0)} - E_n^{(0)}\right)} - \langle n^{(0)}|V|n^{(0)}\rangle \sum_{m\neq n} \frac{|\langle n^{(0)}|V|m^{(0)}\rangle|^2}{\left(E_m^{(0)} - E_n^{(0)}\right)^2}.$$

- so the idea is, for any quantity to do an expansion like this
- NOTE – it only works if the corrections are small and get smaller!
  - This is only true if e is small compared to 1 (see next)

QED

Our interaction term V= $-e\bar{\psi}\gamma^{\mu}A_{\mu}\psi = -e\bar{\psi}\not{A}\psi$

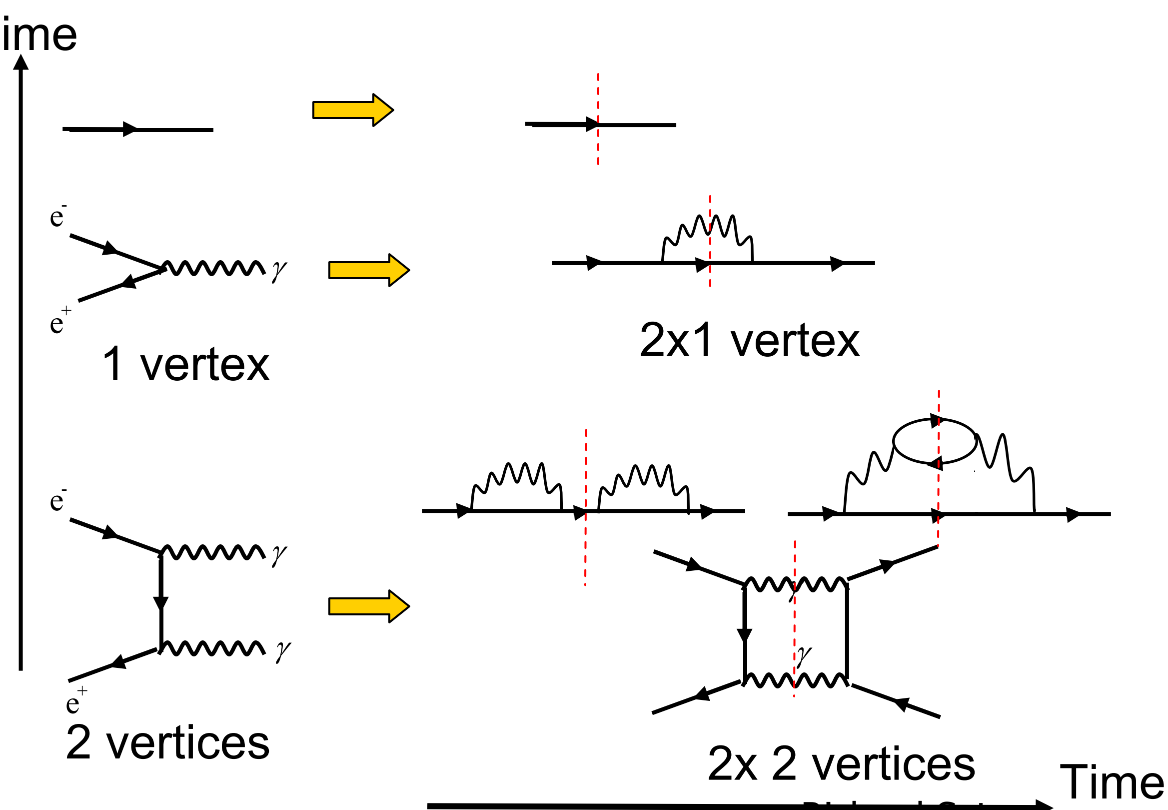
Its a pertubation! I have added an extra constant e

(which turns out, not suprisingly to be the electric charge)

0th order $i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$

1st order $e\bar{\psi}\not{A}\psi$

2nd order $e^2\bar{\psi}\not{A}\psi\bar{\psi}\not{A}\psi$



Now what?

- We started with electrons
- We required the symmetry of local phase invariance – this gave us photons and E&M
- How about other symmetries?
  - can we get the electro-weak theory?
  - can we get the theory of strong interactions?

  - And for the future – maybe quantum gravity? maybe unify all forces????? (no one has succeeded)

More symmetries – groups
(actually semi-simple Lie groups)

- Rotations in 2 D
- U(1) {SO(2)}– 1 generator
  - unitary group
  - SO(2) – special orthogonal group

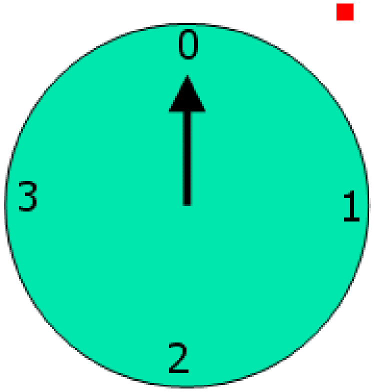
- Rotations in 3 D
- SU(2) {SO(3)} – 3 generators – pauli matrices
  - special unitary group
  - special orthogonal group

- And maybe for later SU(3) – 8 generators (a pain)

Groups - definition

- Objects (A,B,C,…I) with operation "*"
  - Closure $A*B=C$
  - Associative $A*(B*C)=(A*B)*C$
  - There must be Identity $A*I=A$ $B*I=B$ etc
  - There must be inverse $B*(B^{-1})=I$ where $B^{-1}=C$
- Example – Whole numbers under addition
  - …, -2,-1,0,1,2,… , Identity=0, inverse A=-A

Groups – Example with operators



- face of a clock (say with only 0,1,2,3)
  - objects are 0,1,2,3
  - operation is clock addition
  - identity = 0
  - inverse of 2 is 2, inverse of 3 is 1 etc

group multiplication table or "algebra"

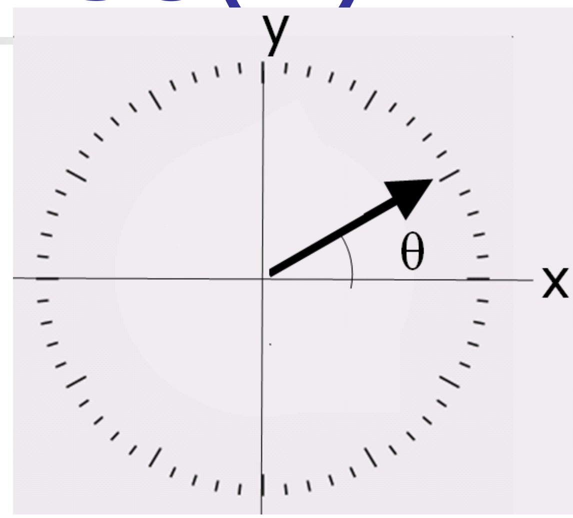
*	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

- How about operators as our group members? rotations?
  - Rotations R(0°), R(90°), R(180°), R(270°)
  - operation is to do the rotations successively i.e
    - R(90)*R(180)|something⟩= R(90)*[R(180)|something⟩]
  - Identity R(0°)
  - Inverse as above

"|something⟩" might be the hand of our clock

*	0	90	180	270
0	0	90	180	270
90	90	180	270	0
180	180	270	0	90
270	270	0	90	180

2D Rotation group – SO(2)



- Continuous group-Infinite number of elements e.g. R_{10}, R_{90}, R_{45} ,….
- Notice these form a group where the "*" is the addition of angles: $R_\alpha R_\beta$(object)=$R_{\alpha\beta}$(object)
- These rotation operators all commute – its "Abelian"
  - $[R_\alpha, R_\beta]=0$
- A way represent such rotations is with the rotation matrix of determinant 1 which can rotate a vector (x,y)
- R_θ is actually the one "generator" of the group. Plug a number in for θ and it "generates" one member of the group

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Note that the matrices (using the * operation for matrices) nicely shows the addition of angles

$$R_\alpha R_\beta = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} =$$

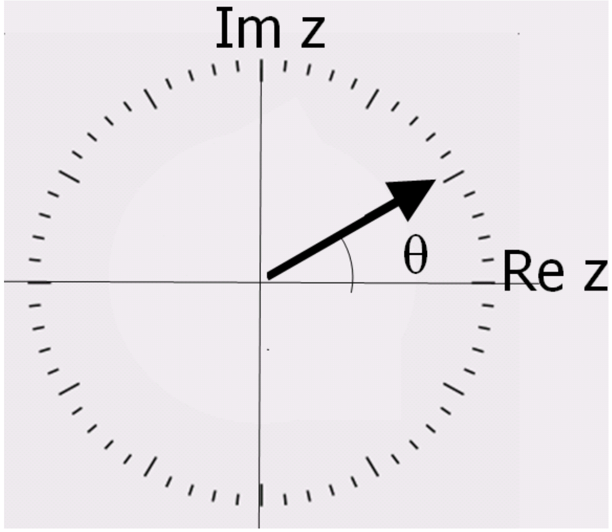
$$\begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = R_{\alpha+\beta}$$

U(1) – another way to think of 2D rotations (almost)

- Lets think of an imaginary number with absolute value one.

  $z=Re(z)+i\ Im(z)$ This can be graphically represented and can be written as $z=\cos\theta+i\sin\theta = e^{i\theta}$

  This can be represented by a 1-D matrix ($e^{i\theta}$) of complex numbers. Again "*" is matrix multiplication $e^{i\alpha}e^{i\beta}=e^{i(\alpha+\beta)}$.



- But what space is this rotating in? Its in a space where one axis is real, the other is imaginary. i.e. complex space – eg. a wavefunction and the rotation can be a phase $e^{i\alpha}$
- This is ALSO a way to represent the group (one to one correspondence of matrices (almost), same multiplication table, same identity, same inverse. U(1) is almost = SO(2) – actually there is a two to one correspondence – this leads to some bizarre behavior of leptons..

3D rotations SO(3)

- Now a 3-D rotation is any rotation of an object (say a book) in 3-space.
- It can be represented by a 3x3 matrix with determinant one.
- it has three generators. *Any* rotation can be made up of a combination of these three

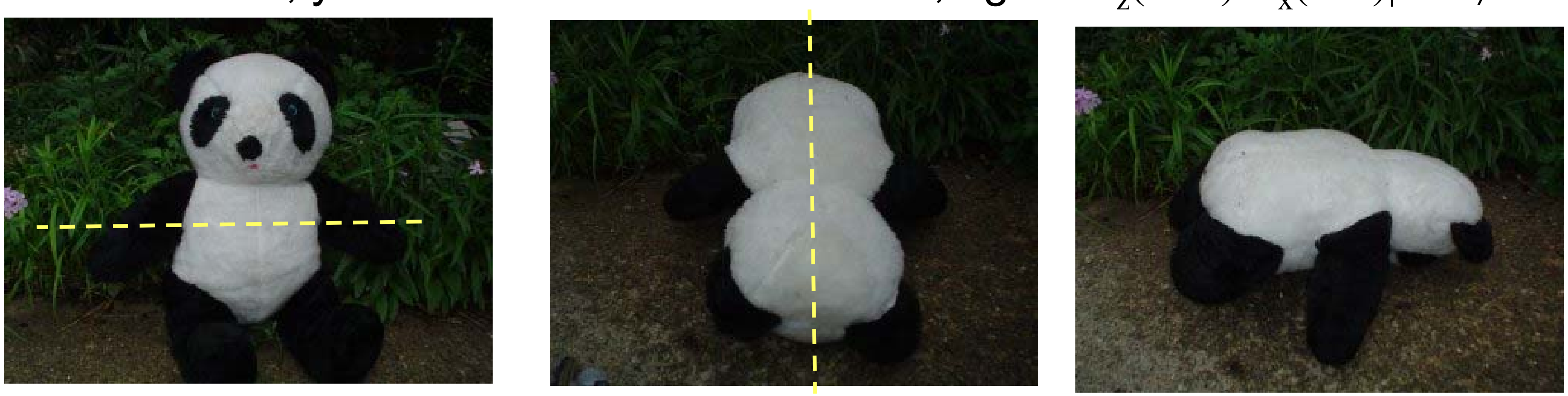
$$R_x(\alpha)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{pmatrix}\quad R_y(\beta)=\begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\quad R_z(\gamma)=\begin{pmatrix}\cos\gamma & \sin\gamma & 0\\ -\sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{pmatrix}$$

You will show in your HW, that the rotation matrices do not commute and follow particular commutation relationships

Rotations in 3-D don’t commute!

Our friend will demonstrate a rotation around the vertical,
followed by one around the horizontal: $R_z(90°)\, R_x(90°)|\mathrm{bear}\rangle$ ⇨

Now, surely, performing the same operations in the *opposite order* will have the same outcome, yes? It in does two dimensions, right? $R_z(90°)\, R_x(90°)|\mathrm{bear}\rangle$ ⇨



Surprise! Rotations about different axes in three dimensions do not commute!
The rotation group in 3-D is non-Abelian. $[R_z(\alpha), R_x(\beta)] \neq 0$

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SU(2)

- SO(2) had a equivalent group of complex matrices U(1), which work on a space of complex numbers (in 1-D)
- Does SO(3) have an equivalent group of complex matrices?
- YES SU(2) – like SO(3) it has three generators, but it works on a complex vector space in 2-D
- They can be represented by 3 2-D matrices
  - Unlike previously I give it to you in a somewhat different form
  - The transformation will be as follows (with similar expressions for y and z) with α_x a number, and 1 is understood as the unit matrix.

$$e^{i\alpha_x\hat{\sigma}_x} = 1 - i\alpha_x\hat{\sigma}_x + \ldots \qquad \vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

the group SU(2) has the same algebra, the same lack of commutations, with the same commutators etc as SO(3), but it lives in some bizarre 2-D complex space

Symmetries AGAIN

- We had a lot of success requiring the lagrangian (physics) to be invariant for transformations under U(1) – i.e. complex numbers of absolute value one
  - $\psi \rightarrow e^{i\theta(x)}\,\psi$
- Does it make sense to require that

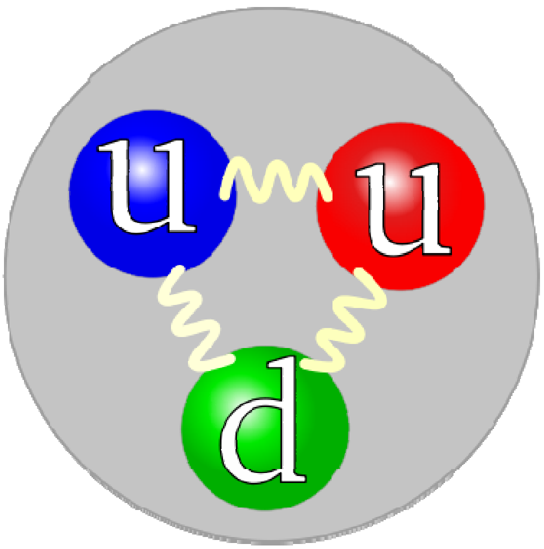
$$\psi(x) \rightarrow e^{i\vec{\alpha}(x)\bullet\vec{\bar{\sigma}}} = (1 - i\vec{\alpha}\bullet\vec{\bar{\sigma}} + \ldots)\psi(x)$$

- but ψ(x) is a complex number, and it needs to be a 2-D vector in some bizarre space, maybe we should think of it as

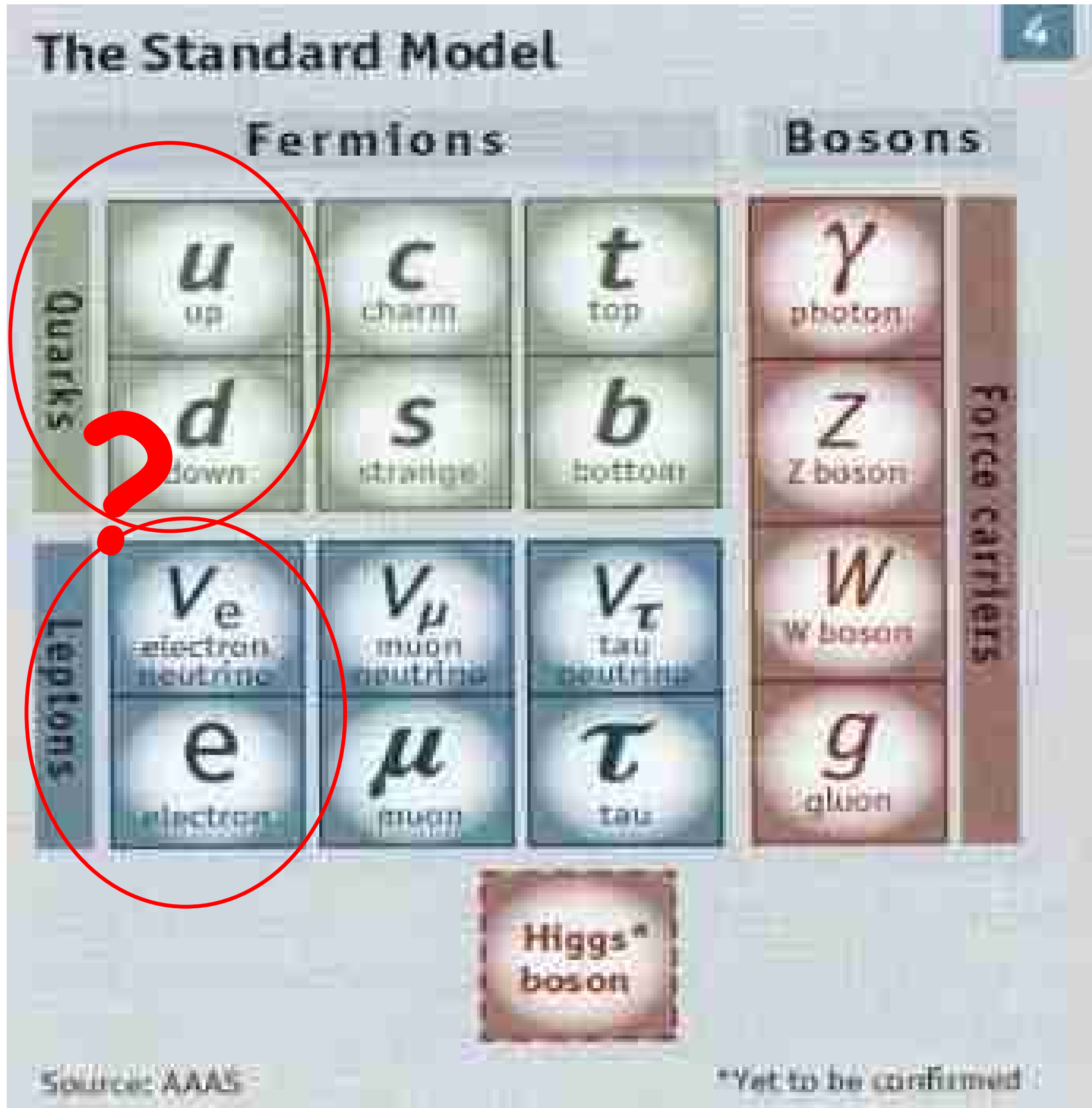
$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$$

and see it corresponds to something in nature

Any doublets?



Proton



Building a theory

Steps

1) choose particles (fermions) and appropriate representation
2) choose gauge group (guess)
3) write down Lagrangian for Kinetic energy of particles
4) check gauge invariance, and add gauge fields (A_μ) to make gauge invariance OK
5) add higgs scalars with higgs potential
6) break symmetry by looking at excitations around minimum of guage potential
7) choose constants to make theory consistent with reality
8) make predictions – is it right?

Step 1) We will just look at the first family, the rest are basically duplicates

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \begin{pmatrix} u \\ d \end{pmatrix}$$

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad (e)_R \quad (u)_R \quad (d_c)_R$$

Note $d_c = d\cos\theta_c + s\sin\theta_c$ where θ_c is the Cabibbo (Kobayashi – Maskawa) angle $d_c = 0.974\, d + 0.2272\, s$

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The higgs mechanism

Interlude

Richard Seto

Spontaneous symmetry breaking

- Which wine glass is yours?
- Or think of a nail sitting on its head

The Higgs Mechanism (and nambu)

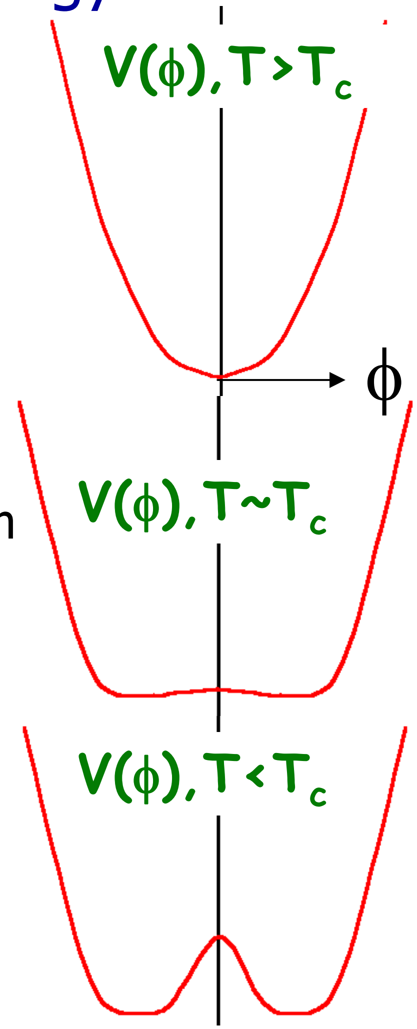
- Invoke a scalar field ϕ with a funny potential energy term

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi - f(T)\frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 - f^2(T)\frac{\mu^4}{4\lambda}$$

$$= \frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi - \frac{\lambda}{4}\left(\phi^2 + f(T)\frac{\mu^2}{\lambda}\right)^2$$

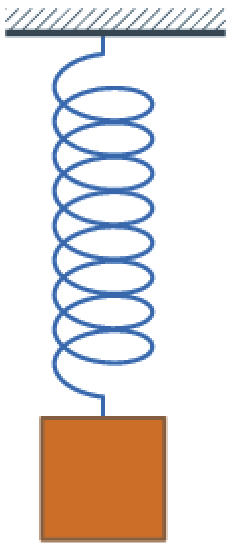
KE of ϕ Potential Scalar field ϕ

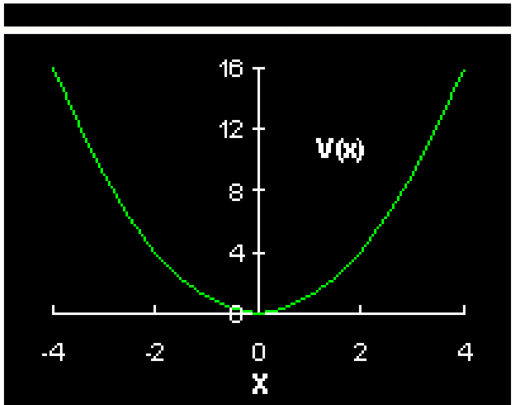
temperature dependent term



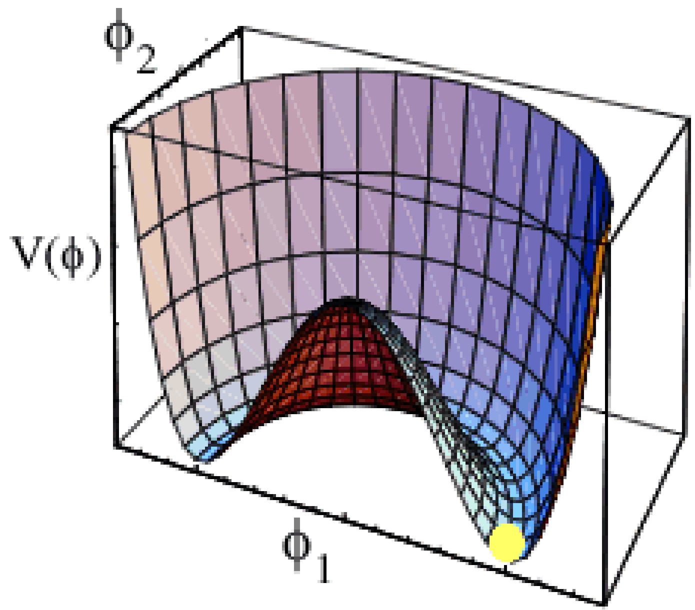
- For High T: $f(T)=+1$ and the lowest energy state is at $\phi = 0$.
- What happens as we lower T? [$f(T)=-1$]
  - Lowest energy state is at $\phi_{min} = \pm\frac{\mu}{\sqrt{\lambda}}$

higgs





The parabolic potential energy of a harmonic oscillator.



tip of hat to nambu

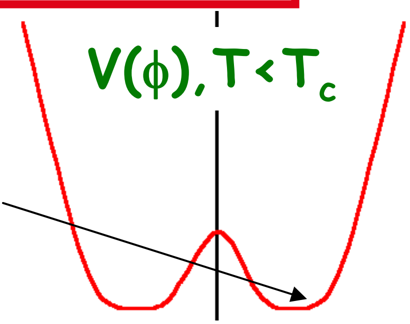
expand around equilibrium point

KE of ψ | Interaction of ϕ with ψ | KE of ϕ | Scalar field ϕ

$$\boxed{\mathcal{L} = i\bar{\psi}\not{\partial}\psi + g\phi\bar{\psi}\psi + \frac{1}{2}\partial_{\mu}\phi\,\partial_{\mu}\phi - \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2}$$

- Start with massless field ψ
- couple ϕ to ψ
- Expand around equilibrium point

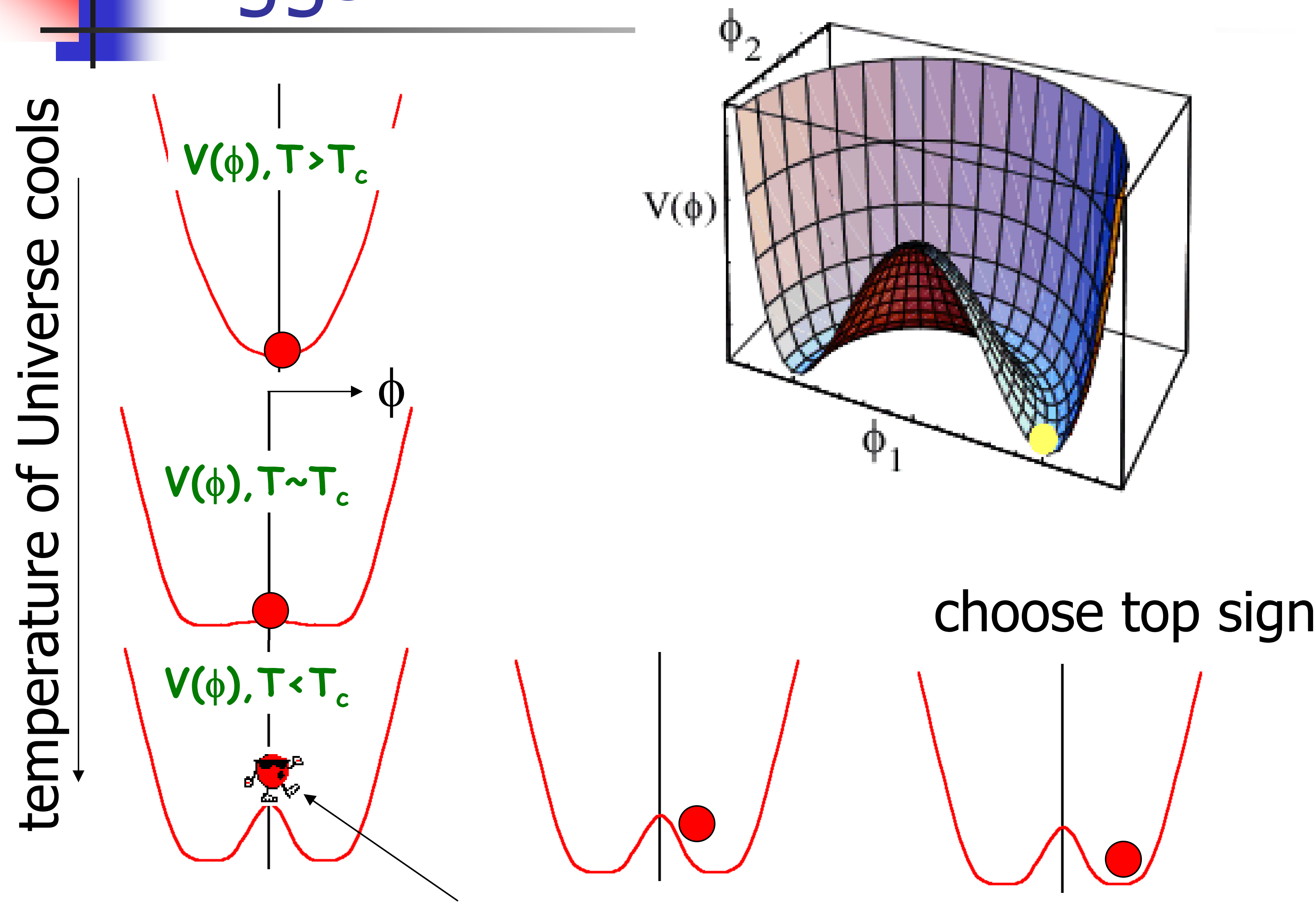
$$\phi' = \phi \pm \frac{\mu}{\sqrt{\lambda}}$$



$$\phi = \phi' \mp \frac{\mu}{\sqrt{\lambda}}$$

spontaneously choose top sign

higgs



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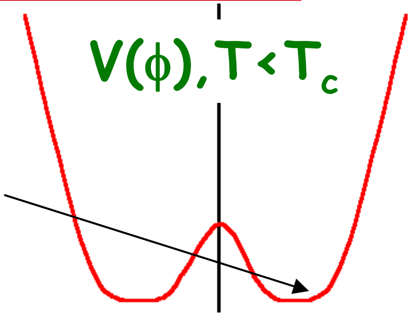
mass from no mass!

KE of ψ — Interaction of ϕ with ψ — KE of ϕ — Scalar field ϕ

$$\boxed{\mathcal{L} = i\bar{\psi}\not{\partial}\psi + g\phi\bar{\psi}\psi + \frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi - \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2}$$

- Start with massless field ψ
- couple ϕ to ψ
- Expand around equilibrium point

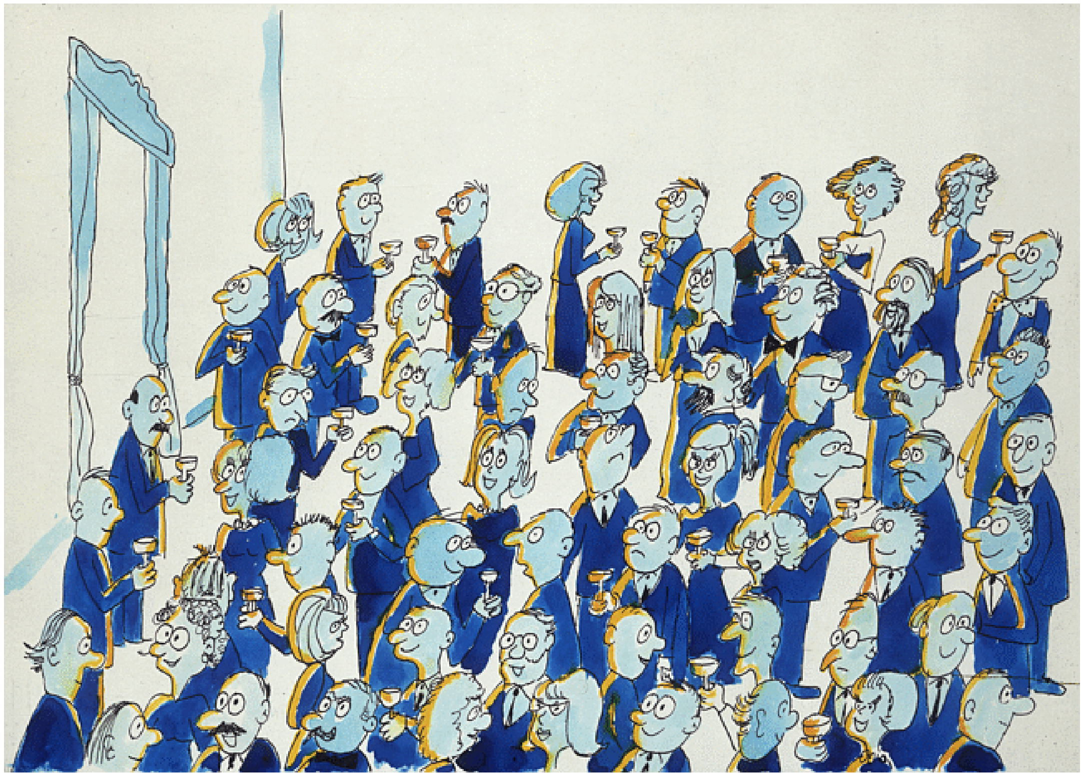
$$\phi' = \phi \pm \frac{m}{\sqrt{\lambda}}$$



$$\mathcal{L}(\phi') = i\bar{\psi}\not{\partial}\psi - g\phi'\bar{\psi}\psi + \frac{g\mu}{\sqrt{\lambda}}\bar{\psi}\psi + \frac{1}{2}\partial_\mu\phi'\,\partial_\mu\phi' - \frac{\lambda}{4}\left(\phi'^2 - \frac{2\mu}{\sqrt{\lambda}}\phi'\right)^2$$

New mass term $\dfrac{g\mu}{\sqrt{\lambda}}$

The higgs mechanism
the higgs field



To understand the Higgs mechanism, imagine that a room
full of physicists quietly chattering is like space filled only with the Higgs field....

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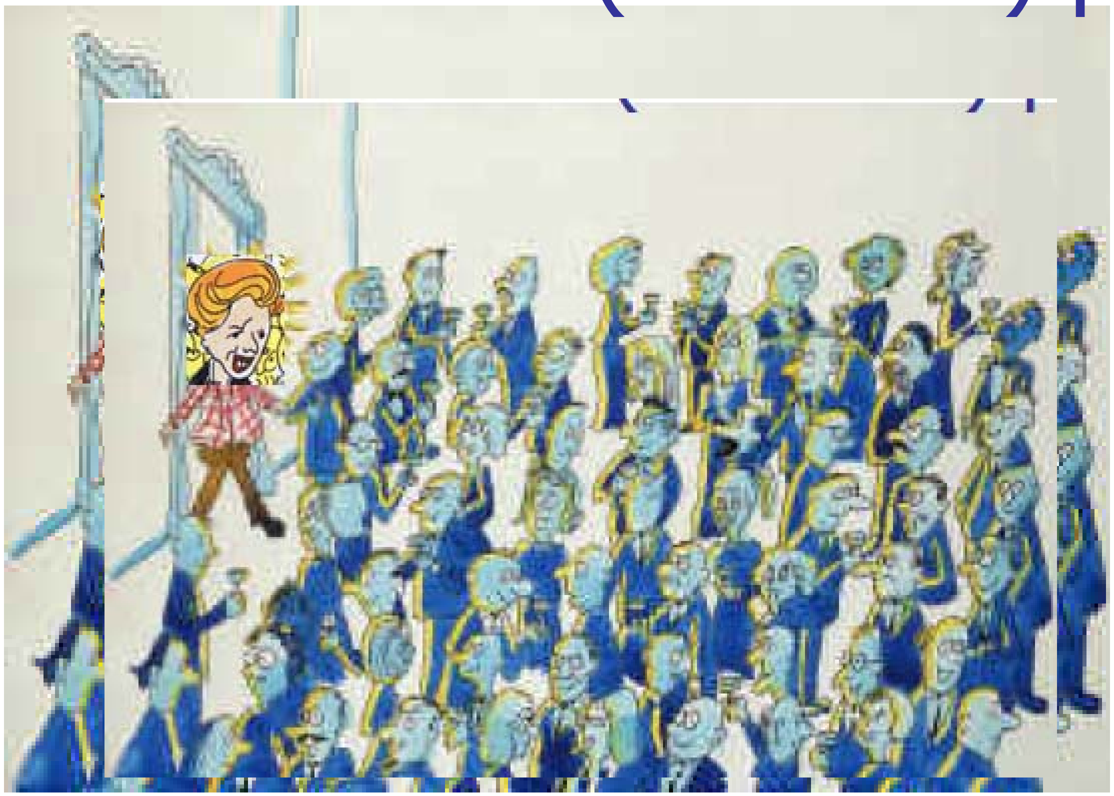
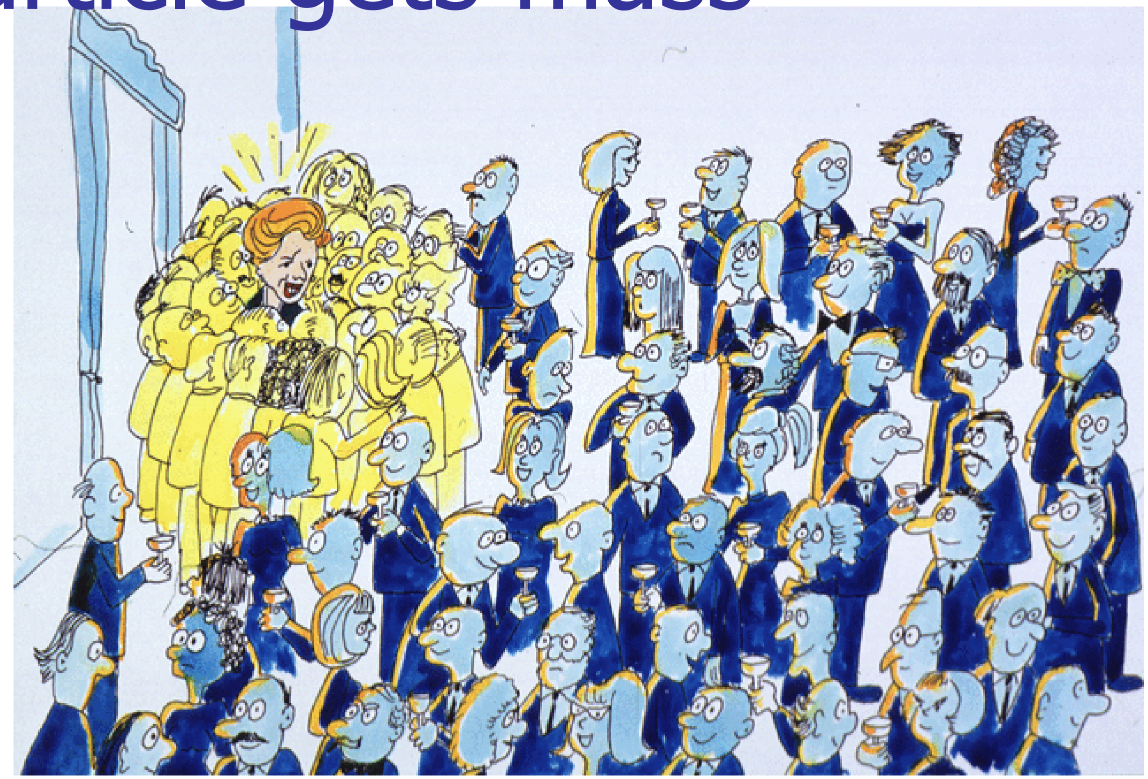
The higgs mechanism

starts out as a bare (massless) particle



... a well known scientist walks in,

The higgs mechanism particle gets mass



creating a disturbance as she moves across the room, and attracting a cluster of admirers with each step

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The higgs mechanism particle is massive

... this increases her resistance to movement, in other words,
he acquires mass, just like a particle moving through the Higgs field ...

The higgs mechanism
the higgs field – gets excited

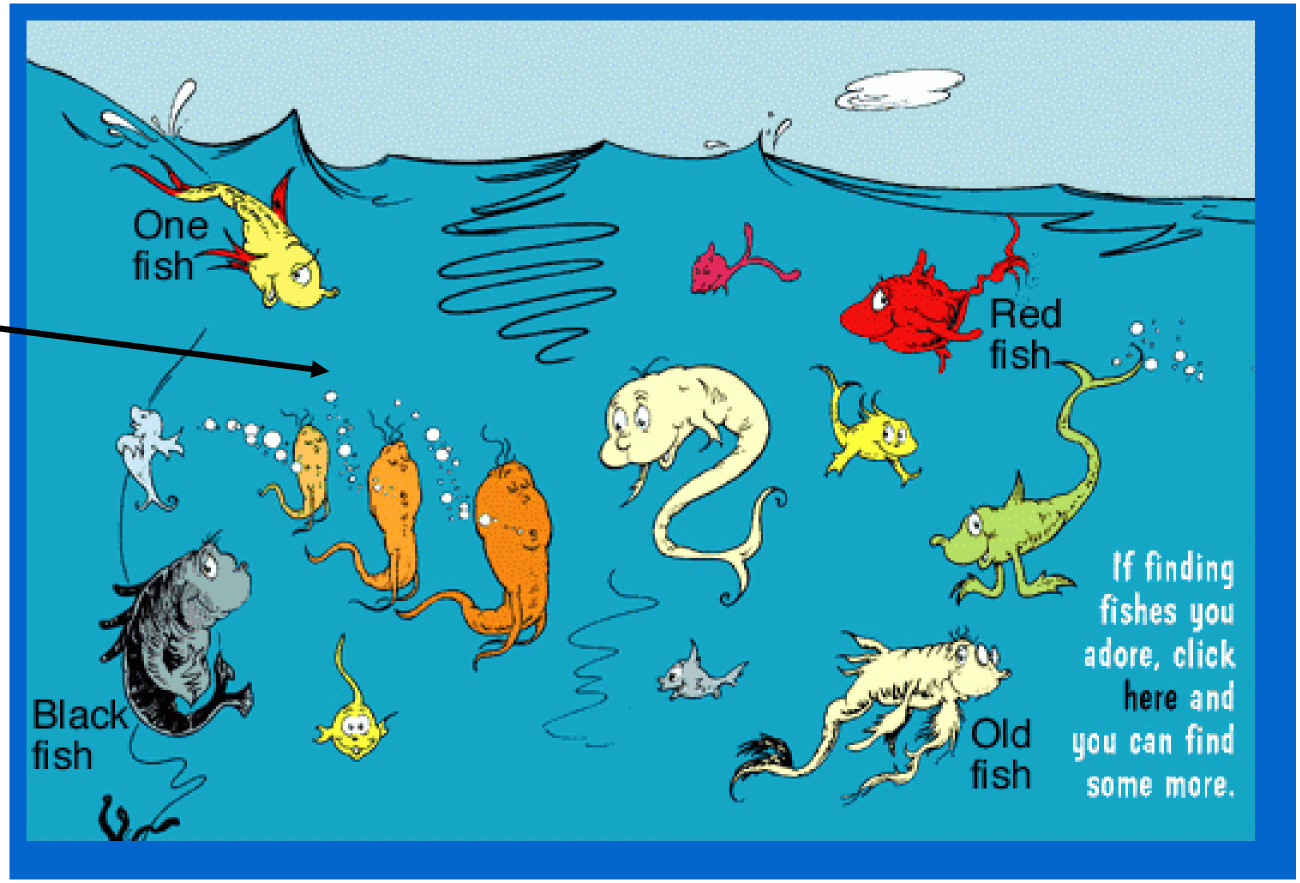
... if a rumour crosses the room ...

The higgs mechanism massive higgs field

... it creates the same kind of clustering, but this time among the scientists themselves. In this analogy, these clusters are the Higgs particles.

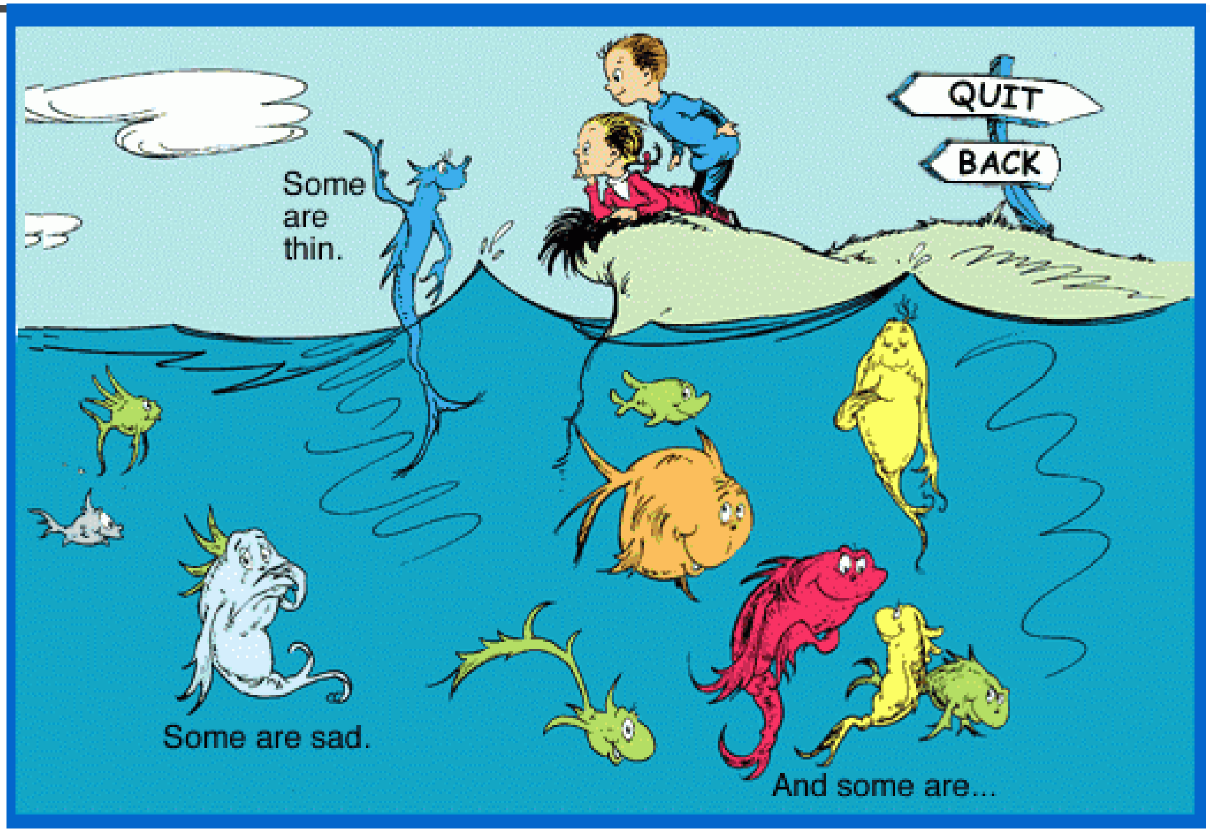
Living in the cold QCD vacuum

The vacuum – perceived to be empty by the general fish population



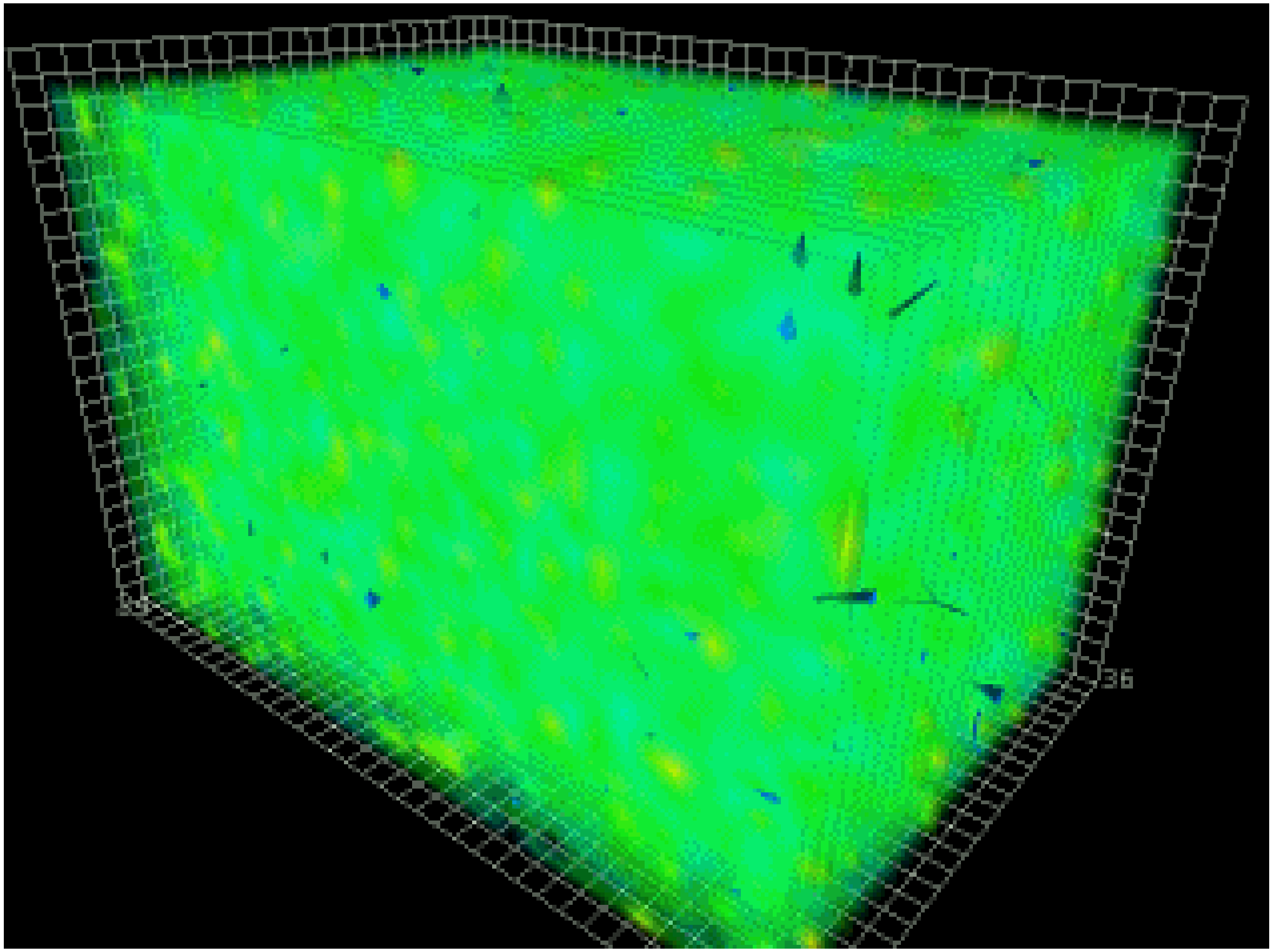
- It is generally believed in the fish population that there is an inherent resistance to motion and that they swim in a "vacuum".

A clever (and crazy idea)



- One clever young fish is enlightened. “The vacuum is complicated and full of water!” he says –” really there is no resistance to motion!”
- “Phooey” say his friends, “we all know the vacuum is empty.”

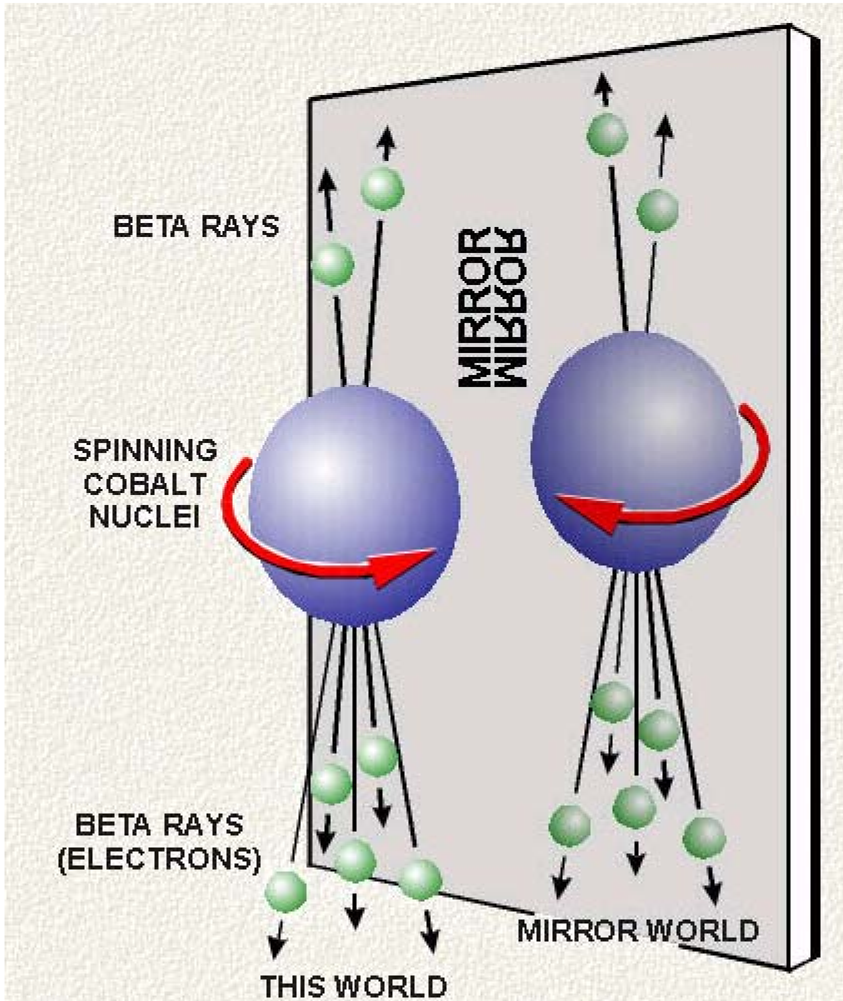
A simulation of the vacuum



4 dimensional Action Density of the vacuum

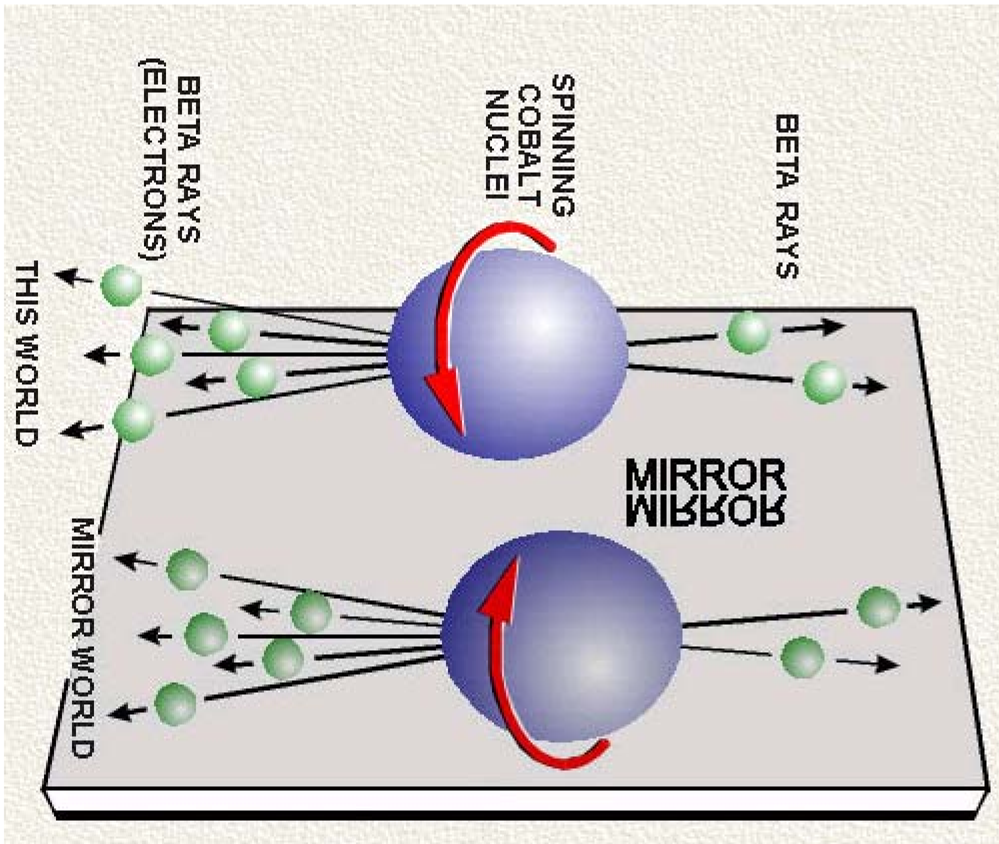
Richard Seto

Weak Force: Parity non-conservation

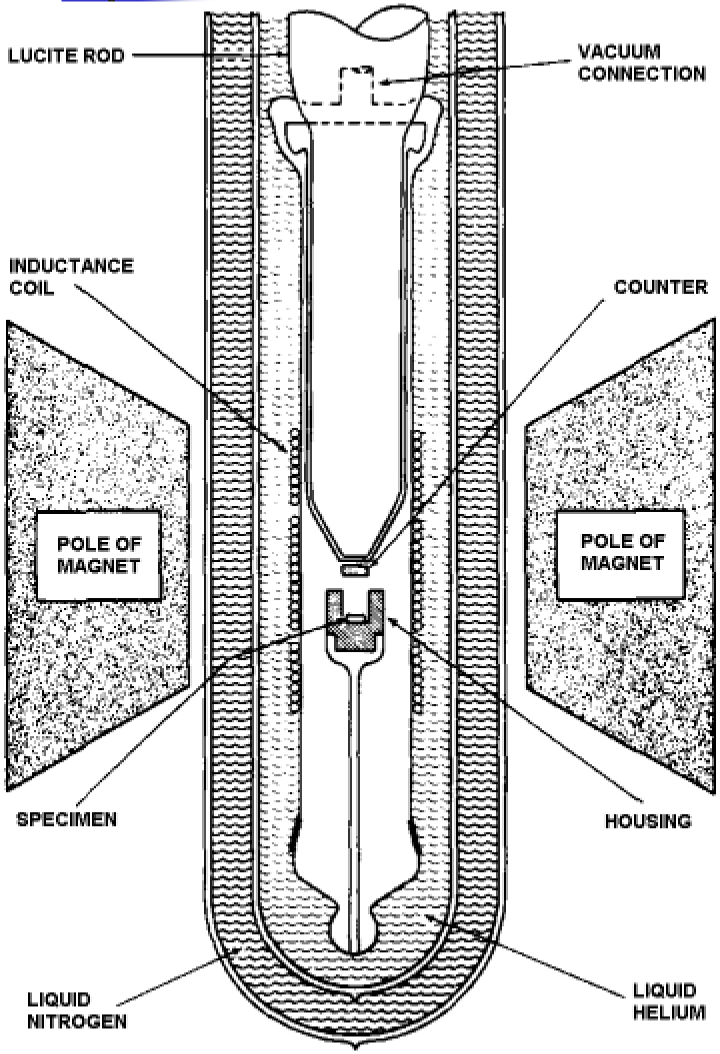


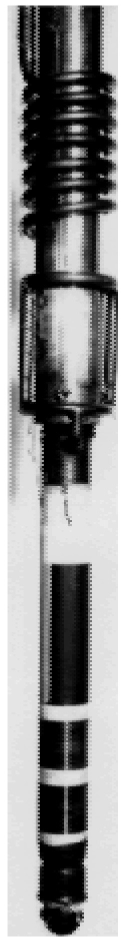
Left ?

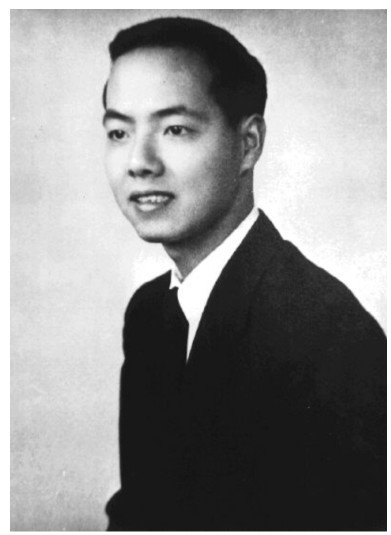
Weak Force: Parity non-conservation



Co-60 experiment 1956







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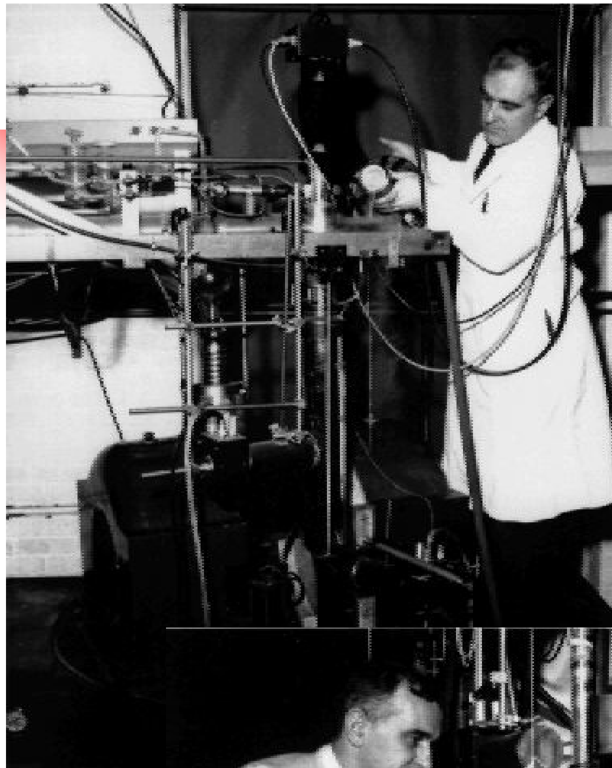

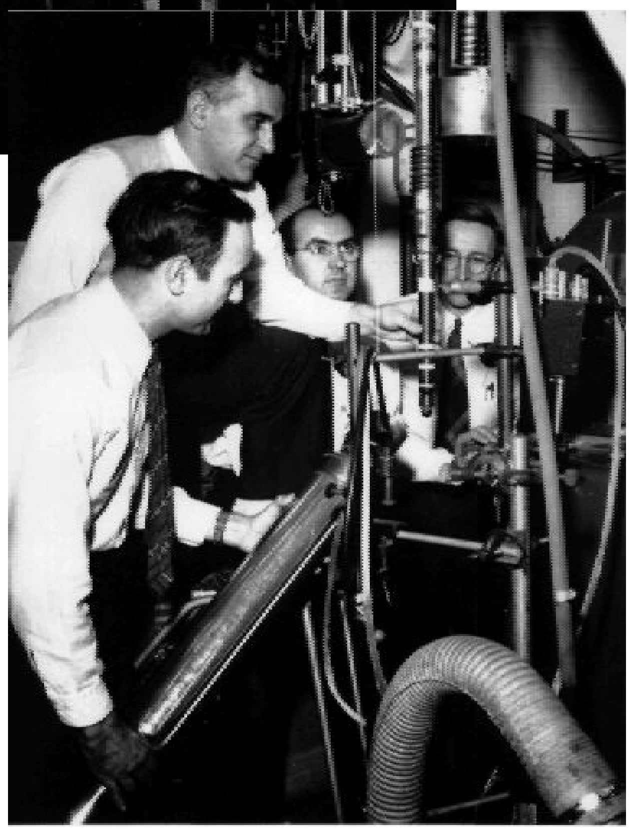

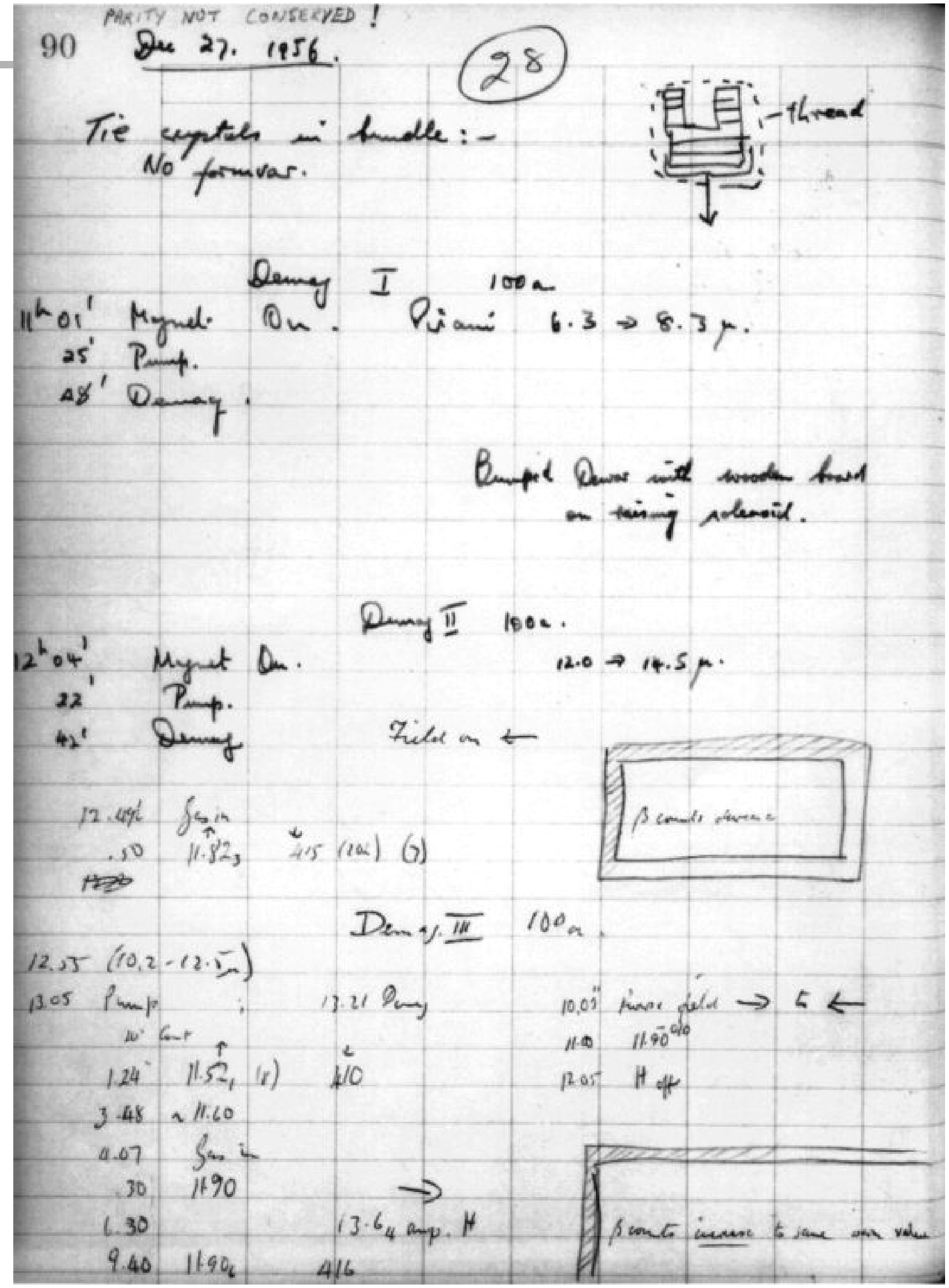

PARITY NOT CONSERVED !

90 Dec 27. 1956. (28)

Tie crystals in bundle :-
No formvar.

— thread

Demag I 100a
11h 01' Magnet On. Pirani 6.3 → 8.3 μ.
25' Pump.
48' Demag.

Bumped Dewar with wooden board on tuning solenoid.

Demag II 100a.
12h 04' Magnet On. 12.0 → 14.5 μ.
22' Pump.
42' Demag

Field on ←

12.48 Gas in
.50 11.82 415 (20) (7)

β counts decrease

Demag. III 100a.
12.55 (10.2 - 12.5 μ)
13.05 Pump 13.21 Demag 10.05 Reverse field → 5 ←
10' Cont 11.00 11.90
1.24 11.52 (8) 410 12.05 H off
3.48 ~11.60
4.07 Gas in
.30 11.90
1.30 13.64 amp. H
9.40 11.904 416

β counts increase to same as value

in muons

[4] Their arguments are as follows: From the He^6 recoil experiment and from Eq. (A-4) of reference 1 one concludes that $(|C_A|^2+|C_A'|^2)/(|C_T|^2+|C_T'|^2)\lesssim\frac{1}{3}$. Hence, by comparing Eq. (16) of reference 3 [see also Eq. (A-6) of reference 1], one concludes that the present large asymmetry is possible only if both conservation of parity and invariance under charge conjugation are violated.

Observations of the Failure of Conservation of Parity and Charge Conjugation in Meson Decays: the Magnetic Moment of the Free Muon*

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LEE and Yang[1-3] have proposed that the long held space-time principles of invariance under charge conjugation, time reversal, and space reflection (parity) are violated by the "weak" interactions responsible for decay of nuclei, mesons, and strange particles. Their hypothesis, born out of the $\tau-\theta$ puzzle,[4] was accompanied by the suggestion that confirmation should be sought (among other places) in the study of the successive reactions

$$\pi^+\rightarrow\mu^++\nu, \quad (1)$$

$$\mu^+\rightarrow e^++2\nu. \quad (2)$$

They have pointed out that parity nonconservation implies a polarization of the spin of the muon emitted from stopped pions in (1) along the direction of motion and that furthermore, the angular distribution of electrons in (2) should serve as an analyzer for the muon polarization. They also point out that the longitudinal polarization of the muons offers a natural way of determining the magnetic moment.[5] Confirmation of this proposal in the form of preliminary results on β decay of oriented nuclei by Wu *et al.* reached us before this experiment was begun.[6]

By stopping, in carbon, the μ^+ beam formed by forward decay in flight of π^+ mesons inside the cyclotron, we have performed the meson experiment, which establishes the following facts:

I. A large asymmetry is found for the electrons in (2), establishing that our μ^+ beam is strongly polarized.

II. The angular distribution of the electrons is given by $1+a\cos\theta$, where θ is measured from the velocity vector of the incident μ's. We find $a=-\frac{1}{3}$ with an estimated error of 10%.

III. In reactions (1) and (2), parity is not conserved.

IV. By a theorem of Lee, Oehme, and Yang,[3] the observed asymmetry proves that invariance under charge conjugation is violated.

V. The g value (ratio of magnetic moment to spin) for the (free) μ^+ particle is found to be $+2.00\pm0.10$.

VI. The measured g value and the angular distribution in (2) lead to the very strong probability that the spin of the μ^+ is $\frac{1}{2}$.[7]

VII. The energy dependence of the observed asymmetry is not strong.

VIII. Negative muons stopped in carbon show an asymmetry (also leaked backwards) of $a\sim-1/20$, i.e., about 15% of that for μ^+.

IX. The magnetic moment of the μ^-, bound in carbon, is found to be negative and agrees within limited accuracy with that of the μ^+.[8]

X. Large asymmetries are found for the e^+ from polarized μ^+ beams stopped in polyethylene and calcium. Nuclear emulsion (as a target in Fig. 1) yields an asymmetry of about half that observed in carbon.

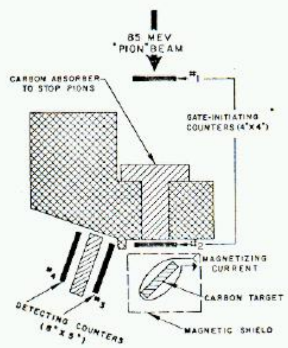

FIG. 1. Experimental arrangement. The magnetizing coil was close wound directly on the carbon to provide a uniform vertical field of 79 gauss per ampere.

The experimental arrangement is shown in Fig. 1. The meson beam is extracted from the Nevis cyclotron in the conventional manner, undergoing about 120° of magnetic deflection in the cyclotron fringing field and about −30° of deflection and mild focusing upon emerging from the 8-ft shielding wall. The positive beam contains about 10% of muons which originate principally in the vicinity of the cyclotron target by pion decay-in-flight. Eight inches of carbon are used in the entrance telescope to separate the muons, the mean range of the "85-"Mev pions being ~5 in. of carbon. This arrangement brings a maximum number of muons to rest in the carbon target. The stopping of a muon is signalled by a fast 1–2 coincidence count. The subsequent beta decay of the muon is detected by the electron telescope 3–4 which normally requires a particle of range >8 g/cm^2(~25-Mev electrons) to register. This arrangement has been used to measure the lifetimes of μ^+ and μ^- mesons in a vast number of elements.[9] Counting rates are normally ~20 electrons/

- In 1956-1957 Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson found a clear violation of parity conservation In the beta decay of cobalt-60. As the experiment was winding down, with doublechecking in progress, Wu informed her colleagues at Columbia of their positive results. Three of them, R. L. Garwin, Leon Lederman, and R. Weinrich modified an existing cyclotron experiment and immediately verified parity violation. They delayed publication until after Wu's group was ready; the two papers appeared back to back.

building a theory

The Weinberg Salam Model

Richard Seto

Steps to building a theory

1) choose particles (fermions) and appropriate representation
2) choose gauge group (guess)
3) write down Lagrangian for Kinetic energy of particles
4) check gauge invariance, and add gauge fields (A_μ) to make gauge invariance OK
5) add higgs scalars with higgs potential
6) break symmetry by looking at excitations around minimum of gauge potential
7) choose constants to make theory consistent with reality
8) make predictions - is it right?

Notation

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

the 4 x 4 matrices $\hat{\gamma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{\gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\gamma}_i = \begin{pmatrix} 0 & \hat{\sigma}_x \\ -\hat{\sigma}_x & 0 \end{pmatrix}$

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = \frac{1}{2}(1-\gamma_5)\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \qquad (e)_R = \frac{1}{2}(1+\gamma_5)(e^-)$$

Note $\quad d_c = d\cos\theta_c + s\sin\theta_c$

where θ_c is the Cabibbo (Kobayashi – Maskawa) angle

$d_c = 0.974\, d + 0.2272\, s$

and e_{ijk} is something called the antisymmetric tensor. $e_{123} = 1$, and if you switch two indices $e_{213} = -1$, switch it again then it will be 1, etc and if indices are the same, its zero.

Intro

First, I will assume that the neutrino has zero mass. We know that is not true because neutrino oscillations have been discovered - this cannot be if the neutrinos all have zero mass. We will also assume that there are no right handed neutrinos, where-as again, neutrino oscillations tells us that they must exist.

The electromagnetic interaction comes from a U(1) symmetry remember (that was the symmetry of the "local phase invariance")

it had an electron (e) , and the electromagnetic field which we will call

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

Now we want to incorporate the weak interaction. We need to figure out the particles, and the then add the gauge field. Experimental evidence shows us that the weak interaction only interacts with left handed particles. $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \quad \begin{pmatrix} u \\ d \end{pmatrix}_L$

The symmetery group which works on a 2 component field is SU(2)

More notation

We will call the gauge field for this part of the interaction

$$F^j_{\mu\nu} = \partial_\mu b^j{}_\nu - \partial_\nu b^j{}_\mu - g\,\epsilon_{jkl}\, b^k{}_\mu b^l{}_\nu$$

the index j=1,2,3 or (x,y,z), is because we will need several of such fields to fix things for us. g is a coupling constant analogous to the electric charge (which for the moment I will NOT call e since it leads to confustion with the actual electron wave function

Don't forget $a_\mu(x)$, $e(x)$ and $\begin{pmatrix} \nu_e\ (x) \\ e^-\ (x) \end{pmatrix}_L$

Step 1 and 2) We will just look at the first family, the rest are basically duplicates

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \begin{pmatrix} u \\ d \end{pmatrix}$$

Group SU (2) x U (1) – these are essentially guesses,
where the U (1) comes from our knowlege of the Electromagnetic interaction
SU (2) comes from our knowlege of the weak interaction.

From Experimental evidence lets write down that the particles that will interact weakly. [Partity Violation Experiments]

$$L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \qquad \begin{pmatrix} u \\ d \end{pmatrix}_L$$

$\overline{L} = (\overline{\nu_e} \;\; \bar{e})$ This $\overline{L}$ is anti – left or right

As I said we will assume there is no right handed neutrino.

Now what about the electromagnetic interaction. the left handed electron and quarks are already chosen, so we better get the right handed parts

$(e)_R \quad (u)_R \quad (d_c)_R \qquad R = (e)_R \qquad \overline{R} = (\bar{e})$

Starting the construct the Lagrangian

Step 3 and 4) Lets write down the Kinetic
terms in the lagrangian We will ignore the quark sector –
it works very simillarly to the lepton sector. The lagrangian must be a scalar so it will be Lorentz invarian

$\mathcal{L}_f = \bar{L}\gamma_\mu \partial_\mu L + \bar{R}\gamma_\mu \partial_\mu R$

Now we cannot have a term like $\bar{L}\gamma_\mu \partial_\mu R$ since writing it out it reads
$(\bar{\nu}_e \;\; \bar{e})(e)_R$ which doesnt make sense (from the matrix notation sense)

The second term is just the same tem we worked with before when we looked at the electron lagragiain. To keep this gauge invarian we have to add a term to make it

$$.\bar{R}\gamma_\mu \partial_\mu R \rightarrow \bar{R}\gamma_\mu \left(\partial_\mu + i\frac{g'}{2} a_\mu\right) R$$

non-Abelian Gauge invariance

Now what about the left handed term? That is a piece which is invariant under SU(2) where the lagrangian is invariant under the

transformation $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \rightarrow \quad e^{i\,\hat{\sigma}\cdot\vec{\theta}(x)} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$ where

$\hat{\sigma} \cdot \vec{\theta}\ (x) = \hat{\sigma}_x\ \theta_x + \hat{\sigma}_y\ \theta_y + \hat{\sigma}_z\ \theta_z$.

To make the lagrangian invariant under this transformation

$.\overline{L}\ \gamma_\mu\ \partial_\mu\ L \rightarrow \overline{L}\ \gamma_\mu \left(\partial_\mu + i\ \frac{g'}{2}\ a_\mu + i\ \frac{g}{2}\ \hat{\sigma}\cdot\vec{b}_\mu\right) L$ where

$\hat{\sigma} \cdot \vec{b}_\mu = \hat{\sigma}_x\ b^1{}_\mu + \hat{\sigma}_y\ b^2{}_\mu + \hat{\sigma}_z\ b^3{}_\mu$

Since we have now added the a and b^j fields we need to add their kineti energies too

$$\mathcal{L}_g = -\frac{1}{4}\ f_{\mu\nu}\ f_{\mu\nu} - \frac{1}{4}\ F^j{}_{\mu\nu}\ F^j{}_{\mu\nu}$$

no explicit mass term?

What about a mass term? We need to make sure that the left and right handed electrons must have the same mass. In addition we want to give mass to the electron but not the neutrino. A term like $m_e \bar{L} R$ is not possible for the same reason that $\bar{L}\, \gamma_\mu\, \partial_\mu\, R$ cannot be a kinetic energy term.

$m\bar{R}\, R = m\, \bar{e}_R\, e_R$ gives a mass to the right handed part of the electron, the left handed part better get the same mass. Does it work?

$m\bar{L}\, L = m\, (\bar{\nu}_e\, \nu_e + \bar{e}_L\, e_L)$ Unfortunately it gives mass to both the electron and the neutrino. So lets forget a mass term. We will generate it using the higgs mechanism. So far

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_g =$$

$$\bar{L}\, \gamma_\mu \left(\partial_\mu + i\, \frac{g'}{2}\, a_\mu + i\, \frac{g}{2}\, \hat{\sigma} \cdot \vec{b}_\mu\right) L + \bar{R}\, \gamma_\mu \left(\partial_\mu + i\, \frac{g'}{2}\, a_\mu\right) R - \frac{1}{4}\, f_{\mu\nu}\, f_{\mu\nu} - \frac{1}{4}\, F^j{}_{\mu\nu}\, F^j{}_{\mu\nu}$$

We will define $D_\mu = \partial_\mu + i\, \frac{g'}{2}\, a_\mu + i\, \frac{g}{2}\, \hat{\sigma} \cdot \vec{b}_\mu$. It turns out that any doublet should have the covariant derivative instead of ordinary derivatives.

Step 5) The higgs mechanism.

Now we need to add the higgs fields and let it interact with the fermion fields. The new twist is that one of our fields is a doublet, i.e. $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$. The only way the higgs field can interact with this thing is to have a higgs doublet as well. Can it also interact with the singlet $(e)_R$. Let's see if we can make this work.

Lets first introduce the higgs doublet (a guess)

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$$ where the numbered fields are real

This has a kinetic energy term $(D_\mu \phi)^\dagger (D_\mu \phi)$. Since we have the a and $\vec{b}$ fields it keeps the lagrangian invariant under gauge transformations.

and a usual potential energy term $V(\phi) = \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$ so

$$\mathcal{L}_s = (D_\mu \phi)^\dagger (D_\mu \phi) - \left(\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2\right)$$

The Lagrangian

What about the interaction term between the fermions and the higgs? Lets try

$$\mathcal{L}_{f-s} = -G_e\left[\overline{R}\left(\phi^\dagger L\right) + \left(\overline{L}\,\phi\right)R\,\right]$$

G_e is yet another constant in addition to g and g'

Notice how we cleverly avoided the problem of coupling the doublet higgs to the singlet electron, by first having the doublet fermion and doublet higs together.

So now we have:

$$\mathcal{L}=\mathcal{L}_f+\mathcal{L}_g+\mathcal{L}_s+\mathcal{L}_{f-s}=$$

$$\overline{L}\,\gamma_\mu\left(\partial_\mu+i\,\frac{g'}{2}\,a_\mu+i\,\frac{g}{2}\,\hat{\sigma}\cdot\vec{b}_\mu\right)L+\overline{R}\,\gamma_\mu\left(\partial_\mu+i\,\frac{g'}{2}\,a_\mu\right)R-\frac{1}{4}\,f_{\mu\nu}\,f_{\mu\nu}-\frac{1}{4}\,F^i{}_{\mu\nu}\,F^i{}_{\mu\nu}$$

$$+(D_\mu\,\phi)^\dagger\,(D_\mu\,\phi)\text{ - }\left(\mu^2\,\phi^\dagger\,\phi+\lambda\left(\phi^\dagger\,\phi\right)^2\right)\quad-G\left[\overline{R}\left(\phi^\dagger L\right)+\left(\overline{L}\,\phi\right)R\,\right]$$

The Lagrangian

$$\mathcal{L}=\mathcal{L}_f+\mathcal{L}_g+\mathcal{L}_s+\mathcal{L}_{f-s}=$$

Kinetic energy of the electron and neutrinos with vector fields to keep gauge invariance

$$\overline{L}\,\gamma_\mu\left(\partial_\mu+i\,\frac{g'}{2}\,a_\mu+i\,\frac{g}{2}\,\hat{\sigma}\cdot\vec{b}_\mu\right)L+\overline{R}\,\gamma_\mu\left(\partial_\mu+i\,\frac{g'}{2}\,a_\mu\right)R$$

$$-\frac{1}{4}\,f_{\mu\nu}\,f_{\mu\nu}-\frac{1}{4}\,F^j_{\mu\nu}\,F^j_{\mu\nu}$$

Kinetic energy of vector fields

$$+(D_\mu\,\phi)^\dagger(D_\mu\,\phi)-\left(\mu^2\,\phi^\dagger\,\phi+\lambda\left(\phi^\dagger\,\phi\right)^2\right)$$

Kinetic and potential energy of scalar (Higgs) field

$$-G_e\left[\overline{R}\left(\phi^\dagger\,L\right)+\left(\overline{L}\,\phi\right)R\,\right]$$

Interaction between electrons and neutrinos with Higgs scalar

Richard Seto

breaking the symmetry

Step 6) break the symmetry. We will assume $\mu^2 < 0$ (low temperatures) and find the minimum

$$\frac{\partial V(\phi)}{\partial \phi} = \mu^2 \phi + 2\lambda (\phi^\dagger \phi)\ \phi = 0$$

There are many solutions to this. Lets choose $\phi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$ were $v = \sqrt{\frac{-\mu^2}{\lambda}}$

Lets check $\phi^\dagger \phi = \frac{v^2}{2} = \frac{-\mu^2}{2\lambda}$

$0 = \left.\frac{\partial V(\phi)}{\partial \phi}\right|_{\phi_0} = \mu^2 \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} + 2\lambda \left(\frac{-\mu^2}{2\lambda}\right) \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so ϕ_0 is a good minimum to expand around.

We want to take a look at small departures (oscillations around ths minimum - i.e. to taylor expand around the minimum).

First we will rewrite ϕ

$$\phi\ (x) = \begin{pmatrix} \phi^+ & (x) \\ \phi^0 & (x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1\ (x) + i\phi_2\ (x) \\ \phi_3\ (x) + i\phi_4\ (x) \end{pmatrix} = e^{\left(i\,\frac{\vec{\xi}(x)\cdot\vec{\sigma}}{2v}\right)} \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} \equiv$$

$$U^{-1}(\xi) \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix}$$

where we have traded the 4 real fields ϕ_{1-4} for the three fields $\vec{\xi}(x)$ and one other one η(x), all real

Note: transforming to unitary gauge (A convenient "gauge", where things are nice)

$$\phi \to \phi' = U(\xi)\phi = \begin{pmatrix} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{pmatrix} \qquad a_\mu \to a_\mu \quad \mathrm{R} \to \mathrm{R} \quad \hat{\sigma}\cdot\vec{\mathrm{b}}_\mu \to \hat{\sigma}\cdot\vec{\mathrm{b}}_\mu{}' \quad \mathrm{L} \to \mathrm{L}' = \mathrm{U}(\xi)\mathrm{L}$$

$$\mathcal{L}_{f-s} = -G_e[\bar{R}(\phi^\dagger L) + (\bar{L}\phi)R] = -G\left(\frac{v+\eta(x)}{\sqrt{2}}\right)\left[(\bar{e}_R)\left((0 \quad 1)\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L\right)\right.$$

$$\left.+\left((\bar{\nu}_e \quad \bar{e})_L \begin{pmatrix}0 \\ 1\end{pmatrix}\right)(e_R)\right] =$$

$$-G_e\left(\frac{v+\eta(x)}{\sqrt{2}}\right)[(\bar{e}_R)(e_L) + (\bar{e}_L)(e_R)] = \qquad \text{(see aside)}$$

$$-G_e\left(\frac{v+\eta(x)}{\sqrt{2}}\right)[(\bar{e})(e)] = -G\frac{v}{\sqrt{2}}\overline{e(x)}e(x) - G\frac{\eta(x)}{\sqrt{2}}\overline{e(x)}e(x)$$

electron mass term!

so we have now generated a mass for the electron!

$$m_e = \frac{G_e v}{\sqrt{2}}$$

Aside: $\bar{e}\,e = (\bar{e}_R + \bar{e}_L)(e_R + e_L) = \bar{e}_R e_R + \bar{e}_R e_L + \bar{e}_L e_R + \bar{e}_L e_L = \bar{e}_R e_L + \bar{e}_L e_R + 0$ since [remember $\bar{e}_R$ is actually a left handed object]

$$\bar{e}_R e_R = \bar{e}(1-\gamma_5)(1+\gamma_5)e = \bar{e}(1-\gamma_5\gamma_5)e = 0 \qquad \text{since } \gamma_5\gamma_5 = 1$$

The second term represents a coupling between the electron and the field $\eta(x)$

Now that we have had some success is generating the electron mass, lets charge on to the rest of the terms in the lagrangian and expand in the new variables and see what we get. I will not show the algebra, since there is a lot of it.

The W's and Z's appear

$$\mathcal{L}_s = (D_\mu \phi)^\dagger (D_\mu \phi) - \Big(\mu^2 \phi^\dagger \phi + \lambda\big(\phi^\dagger \phi\big)^2\Big) =$$

$$1/2\,(\partial_\mu \eta)(\partial_\mu \eta) - \frac{1}{2} m_\eta{}^2 \eta^2 + \frac{1}{2} m_W{}^2\Big(|W_\mu{}^+|^2 + |W_\mu{}^-|^2\Big) + \frac{1}{2} m_Z{}^2 |Z_\mu{}^0|^2 + \text{interaction terms between the higgs and the vector bosons}$$

where we have defined the charged boson fields

$$W_\mu{}^+ \equiv \frac{1}{\sqrt{2}}\left(b_\mu{}^1 - i b_\mu{}^2\right) \qquad W_\mu{}^- \equiv \frac{1}{\sqrt{2}}\left(b_\mu{}^1 + i b_\mu{}^2\right)$$

and the neutral boson fields

The Weinberg angle sin θ_W

$$Z_\mu^0 \equiv \frac{\left(-g' a_\mu + g b_\mu^3\right)}{\sqrt{g^2+g'^2}} \qquad A_\mu \equiv \frac{\left(g a_\mu + g' b_\mu^3\right)}{\sqrt{g^2+g'^2}}$$ with masses

$$m_\eta = \sqrt{-2\,\mu^2} \qquad m_W = \frac{gv}{2} \qquad m_Z = m_W \sqrt{1+\left(\frac{g'}{g}\right)^2} \quad \text{and} \quad m_A = 0$$

So now we have 2 massive charged bosons $W^\pm$, a massive neutral boson Z^0, and the mass neutral boson A as well as a higgs with $m_\eta = \sqrt{-2\,\mu^2}$

Lets define $\tan\theta_W \equiv \frac{g'}{g}$ so $\sqrt{g^2+g'^2} = \frac{g}{\cos\theta_W} = \frac{g'}{\sin\theta_W}$ and we can write

$$Z_\mu^0 = -a_\mu \sin\theta_W + b_\mu^3 \cos\theta_W \qquad A_\mu = a_\mu \cos\theta_W + b_\mu^3 \sin\theta_W$$

$$m_Z = \frac{m_W}{\cos\theta_W}$$

Interactions

Interactions

for the fermions $\mathcal{L}_f = \bar{L}\gamma_\mu \partial_\mu L + \bar{R}\gamma_\mu \partial_\mu R = \mathcal{L}_f^{\text{charged}} + \mathcal{L}_f^{\text{neutral}}$

$\mathcal{L}_f^{\text{charged}} = \frac{-g}{\sqrt{2}}(\bar{\nu}_L \gamma^\mu e_L W_\mu{}^+ + \bar{e}_L \gamma^\mu \nu_L W_\mu{}^-)$ So the charged current weak interaction only see left handed stuff!

we get that $G_{\text{Fermi}} = \frac{\sqrt{2}\, g^2}{8\, m_W{}^2}$ where $G_{\text{Fermi}} \sim \frac{10^{-5}}{m_p{}^2}$ is the Fermi coupling constant which can be measured in β decay

$\mathcal{L}_f^{\text{neutral}} = q\, \bar{e}\, \gamma^\mu A_\mu\, e - \frac{g}{2\cos\theta_W}\Big[\bar{\nu}_L \gamma^\mu \nu_L Z_\mu{}^0 + 2$
$\sin^2\theta_W\, \bar{e}_R \gamma^\mu e_R Z_\mu{}^0 + \left(2\sin^2\theta_W - 1\right)\bar{e}_L \gamma^\mu e_L Z_\mu{}^0\Big]$

Notice first that the neutral part of the weak interaction DOES see the right handed electron.

electric charge

We have now found that q, the electric charge $\left(1.602\, e^{-19} \text{ Coulombs}\right)$

$$q \equiv \frac{gg'}{\sqrt{g^2+g'^2}} = g \sin\theta_W$$

SO now we have a standard QED coupling of the photons to charged electrons. The photons do not couple to the neutrinos. We also have

a neutral weak current.

What do we know?

Predictions and comparisons with data

first what constants have we introduced?

v, μ, g, g', G_e

What do we know?

$m_e = 0.5\ \frac{\text{MeV}}{c^2}$ q - actually we know $\frac{q^2}{4\pi\hbar c} = \frac{1}{137}$ $G_{\text{Fermi}} = \frac{10^{-5}}{m_p{}^2}$ where

$m_p = 938\ \frac{\text{MeV}}{c^2}$

we have now

$m_e = \frac{G_e\, v}{\sqrt{2}}$ $G_{\text{Fermi}} = \frac{\sqrt{2}\, g^2}{8\, m_w{}^2}$ $m_W = \frac{gv}{2}$ $m_Z = \frac{m_W}{\cos\theta_w}$

$g = \frac{q}{\sin\theta_w}$ $g' = \frac{q}{\cos\theta_w}$

$$m_W = \left(\frac{q^2}{4\sqrt{2}\ G_{\text{Fermi}} \sin^2\theta_W}\right)^{1/2} = \frac{37.3}{\sin\theta_W}\ \text{GeV}$$

we see then that $G_{\text{Fermi}} \sim \dfrac{q^2}{{m_W}^2}$ The smallness of the weak interaction is due to the large mass of the W boson.

We can solve for v (the expectation value of the vacuum!)

$$\frac{v}{\sqrt{2}} = (\sqrt{8}\ G_{\text{Fermi}})^{-\frac{1}{2}} \sim 174\ \text{GeV}$$

$$G_e = \frac{\sqrt{2}\ m_e}{v} = 3 \times 10^{-6}$$

We need a measurment of θ_W

this comes from the scattering of electrons by neutrinos.

$\sin^2\theta_W = 0.23$

Giving $m_W = 78\ \dfrac{\text{GeV}}{c^2}$ $\quad m_Z = 90\ \dfrac{\text{GeV}}{c^2}$

Experiment confirms theory (to some extent)

After this was done, experimentalistss were told to look in this mass region and they found:

$$m_W = 80.40 \ \frac{\text{GeV}}{c^2}$$

$$m_Z = 91.19 \ \frac{\text{GeV}}{c^2}$$

(note radiative corrections alters the predictions for the vector boson masses)

So with 3 inputs m_e, $\alpha = \frac{1}{137}$, G_F, and θ_W

we figured out m_W, m_Z, v, G_e, g, g'

Unfortunately the mass of the higss $m_\eta = \sqrt{-2\,\mu^2}$ is not connected to anything so we don't know what it is.