Event-by-Event Averages In Heavy Ion Collisions

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625 tracks, 733 EMcal clusters in $\Delta\phi = \pi$, $\delta\eta = 0.70$

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Some Favorite Statistics

•In the theory of probability and statistics, a statistic is a quantity computed entirely from the sample, i.e. a statistic is any function of the observed sample values.

•Two of the most popular statistics are the sum and the average:

$$S_n \equiv \sum_{i=1}^{n} x_i \tag{1}$$

$$\bar{x}_{(n)} \equiv \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n} S_n \tag{2}$$

where the x_i are n samples from a the same population or probability density function, $f(x)$.

• From the theory of mathematical statistics, the probability distribution of a random variable S_n, which is itself the sum of n independent random variables with a common distribution $f(x)$:

$$S_n = x_1 + x_2 + \cdots + x_n \tag{3}$$

is given by $f_n(x)$, the n-fold convolution of the distribution $f(x)$:

$$f_n(x) = \int_0^x dy\, f(y)\, f_{n-1}(x-y) \qquad . \tag{4}$$

The mean, $\mu_n = \langle S_n \rangle$, and standard deviation, σ_n, of the n-fold convolution obey the familiar rule

$$\mu_n = n\mu \qquad \sigma_n = \sigma\sqrt{n} \quad , \tag{5}$$

where μ and σ are the mean and standard deviation of the distribution $f(x)$.

• A complementary case is that of a random variable Z_n, which is the sum of n random variables with a common distribution $f(x)$—which are themselves 100% correlated—for example:

$$Z_n = x + x + \cdots + x = nx \qquad . \tag{6}$$

This is just a scale transformation. The behavior of the mean and the standard deviation for a scale transformation is $\mu \to n\mu$, $\sigma \to n\sigma$, which is quite different than the behavior of the standard deviation under convolution (Eq. 5).

Some Favorite Functions

The Gamma distribution is an example of a probability density function (pdf) which has particularly simple properties under convolutions and scale transformations. The Gamma distribution is a function of a continuous variable x and has paramters p and b

$$f(x) = f_\Gamma(x,p,b) = \frac{b}{\Gamma(p)}(bx)^{p-1}e^{-bx} \tag{7}$$

where

$$p > 0, \quad b > 0, \quad 0 \leq x \leq \infty$$

$\Gamma(p) = (p-1)!$ if p is an integer, and $f(x)$ is normalized, $\int_0^\infty f(x)dx = 1$. The mean and standard deviation of the distribution are

$$\mu \equiv \langle x \rangle = \frac{p}{b} \qquad \sigma \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{\sqrt{p}}{b} \qquad \frac{\sigma^2}{\mu^2} = \frac{1}{p} \qquad . \tag{8}$$

The n-fold convolution of the Gamma distribution (Eq. 7) is simply given by the function

$$f_n(x) = \frac{b}{\Gamma(np)}(bx)^{np-1}e^{-bx} = f_\Gamma(x,np,b) \tag{9}$$

i.e. $p \rightarrow np$ and b remains unchanged. Note that the mean and standard deviation of Eq. 9

$$\mu_n = \frac{np}{b} \qquad \sigma_n = \frac{\sqrt{np}}{b} \qquad \frac{\sigma_n}{\mu_n} = \frac{1}{\sqrt{np}} \tag{10}$$

when compared to Eq. 8 explicitly obey Eq. 5. The result of a scale transformation $x \rightarrow nx$ for a Gamma distribution (Eq. 7) is simply $b \rightarrow b/n$, with p remaining unchanged. **To summarize, the n-th convolution of the Gamma distribution $f_\Gamma(x,p,b)$ is $f_\Gamma(x,np,b)$; the scale transformation $x \rightarrow nx$ of $f_\Gamma(x,p,b)$ is $f_\Gamma(x,p,b/n)$.**

The p_T distribution is a Gamma Distribution

The principal advantage of the Gamma distribution for the present problem is that it is one of the standard representations of the inclusive single particle p_T distribution:

$$\frac{d\sigma}{p_T dp_T} = b^2 e^{-bp_T} \quad (11)$$

$$\frac{d\sigma}{dp_T} = b^2 p_T e^{-bp_T} \quad . \quad (12)$$

Clearly, Eq.s 11, 12 represent a Gamma distribution with $p = 2$, $\langle p_T \rangle = 2/b$, where typically $b = 6$ (GeV/c)$^{-1}$ for p-p collisions. The 'inverse slope parameter' $1/b$ is sometimes referred to as the 'Temperature parameter'.

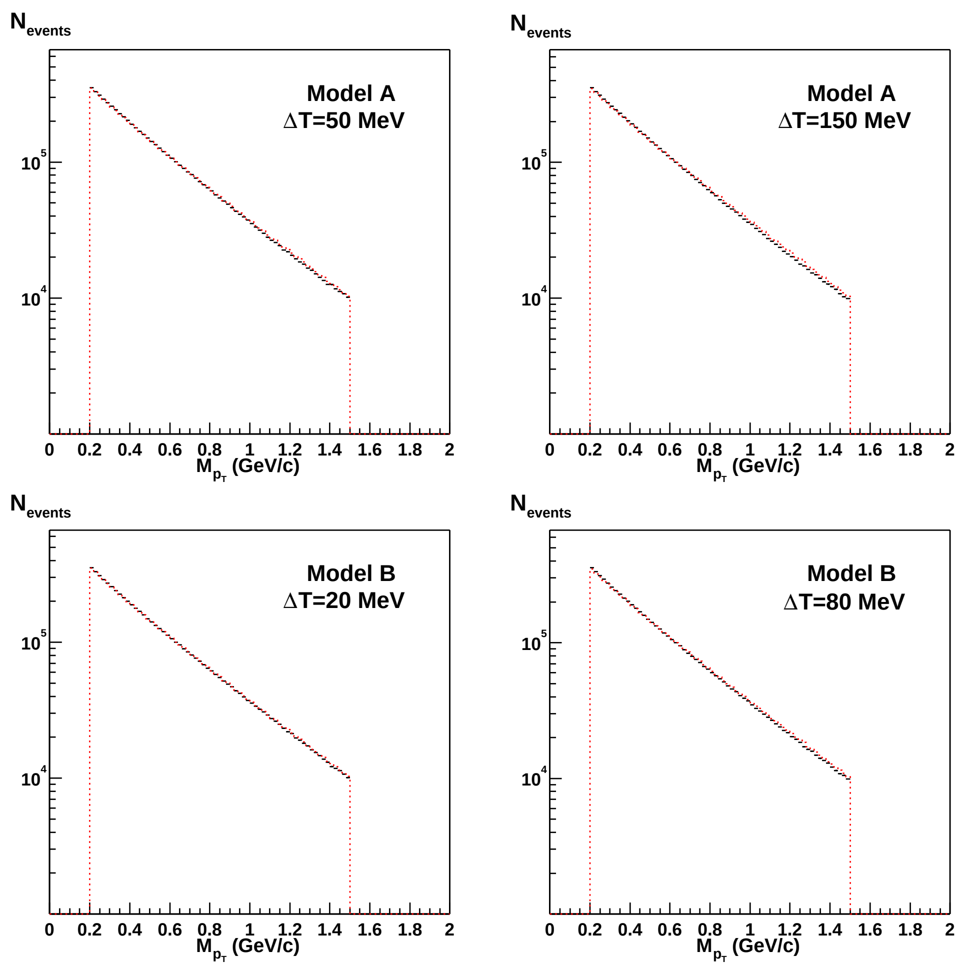

Figure 1: PHENIX inclusive p_T **Not** M_{p_T} distribution for 5% most central collisions(black). Red are 2-component models to be discussed later. Note the limited range of p_T used.

Event by Event Distribution
It's a Gamma Distribution not a Gaussian

$$M_{p_T} = \overline{p_T}_{(n)} = \frac{1}{n}\sum_{i=1}^{n} p_{T_i} = \frac{1}{n} E_{Tc}$$

$$E_{Tc} = \sum_{i=1}^{n} p_{T_i}$$

Analytical formula for statistically independent emission

For statistical independent emission an analytical formula for the distribution in M_{p_T} can be obtained. It depends on the 4 semi-inclusive parameters $\langle n \rangle$, $1/k$, b and p which are derived from the quoted means and standard deviations of the semi-inclusive p_T and multiplicity distributions. The result is in excellent agreement with the NA49 Pb+Pb-central measurement.

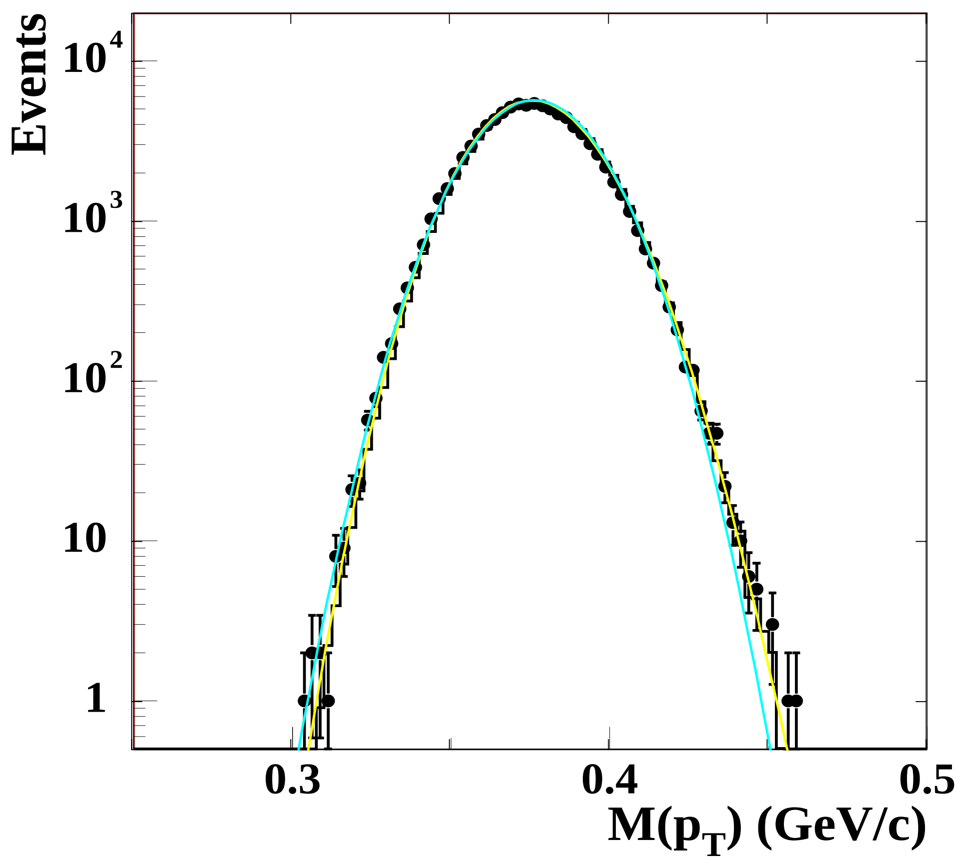

Figure 2: Gamma Distribution for M_{p_T} (yellow) compared Gaussian with same μ and σ (light blue) for NA49 measurement (filled points) and mixed event distribution (histogram).

see M. J. Tannenbaum, Phys. Lett. B498, 29 (2001)

Gamma Distribution for PHENIX central M_{p_T}

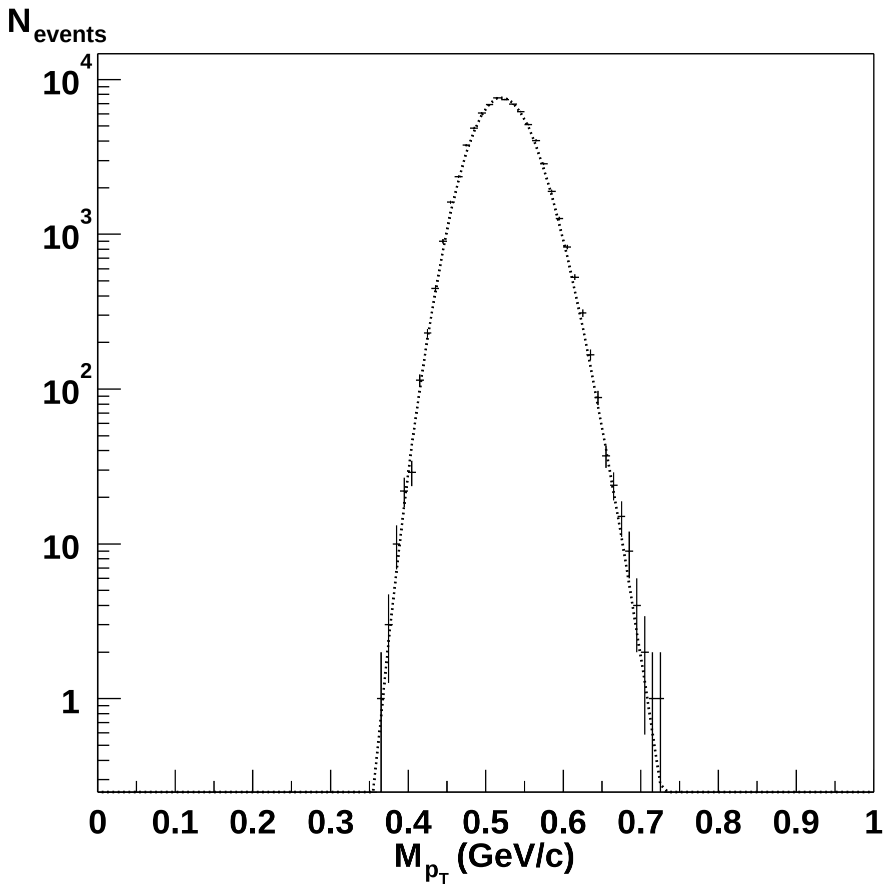

Figure 3: PHENIX M_{p_T} data for Au+Au central 5% at $\sqrt{s_{NN}}$ = 130 GeV compared to Gamma Distribution calculation from semi inclusive parameters.

$$f(y) = \sum_{n=n_{\min}}^{n_{\max}} f_{\rm NBD}(n, 1/k, \langle n\rangle)\, f_{\Gamma}(y, np, nb) \qquad . \tag{13}$$

This is the analytical formula for M_{p_T} assuming NBD distributed event-by-event multiplicity, with Gamma distributed semi-inclusive p_T spectrum, assuming statistically independent emission of particles on each event. It depends on the 4 semi-inclusive paramters $\langle n\rangle$, σ_n, $\langle p_T\rangle$, σ_{p_T}, where

$$\frac{1}{k} = \frac{\sigma_n^2}{\langle n\rangle^2} - \frac{1}{\langle n\rangle}$$

$$p = \frac{\langle p_T\rangle^2}{\sigma_{p_T}^2} \qquad b = \frac{\langle p_T\rangle}{\sigma_{p_T}^2}$$

PHENIX M_{p_T} vs. Centrality
Now Gamma Distribution Shape is Obvious

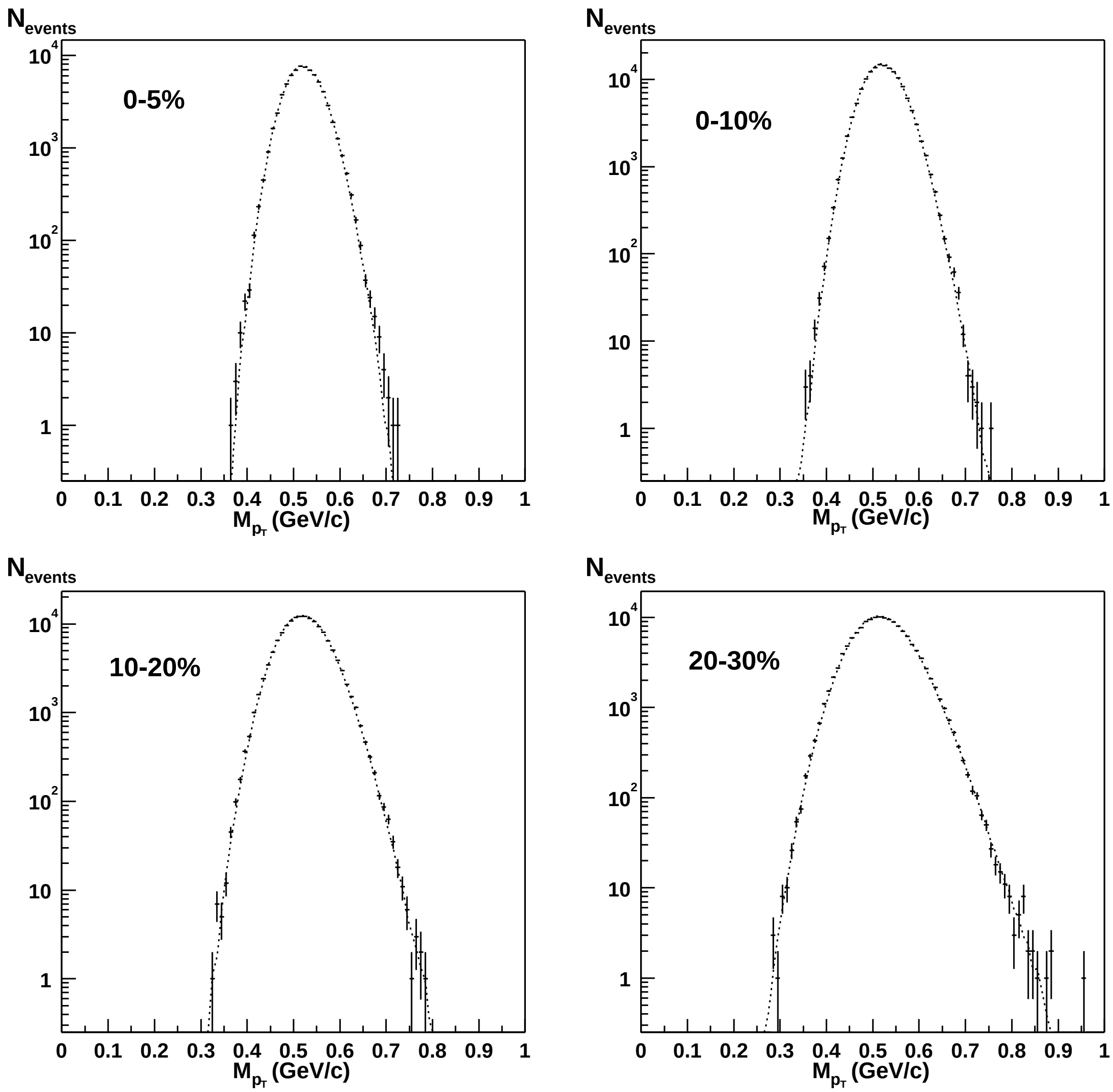

Figure 4: PHENIX M_{p_T} data for Ai+Au at $\sqrt{s_{NN}} = 130$ GeV vs centrality. The dotted curves are mixed event distributions used as the random baseline.

- Use Mixed Events as a random baseline reference since analytical formula doesn't work in general.
- For Mixed Events **must use exactly the same n distribution as the data and match the inclusive $\langle p_T \rangle$ to high precision.**
- Data indicate very small, if any, non random effect **How to Quantify?**

Most Groups Use Moments (fortunately just μ and σ)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\langle \bar{x} \rangle = \langle x \rangle \equiv \mu$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$\sigma_{\bar{x}}^2 = \langle \bar{x}^2 \rangle - \langle \bar{x} \rangle^2$$

•For Statistically Independent Emission

$$\sigma_{\bar{x}}^2 = \left\langle \frac{\sigma_x^2}{n} \right\rangle$$

Typical Measures of Sensitivity

$$\frac{\text{Measured -Random}}{\text{Random}}$$

$$\left(\frac{\sigma_{\bar{x}}^2}{\mu^2} - \frac{1}{n}\frac{\sigma_x^2}{\mu^2}\right) / \frac{1}{n}\frac{\sigma_x^2}{\mu^2}$$

$$\left(\frac{\sigma_{\bar{x}}}{\mu} - \frac{1}{\sqrt{n}}\frac{\sigma_x}{\mu}\right) / \frac{1}{\sqrt{n}}\frac{\sigma_x}{\mu} = F$$

•For small effects these measures are equivalent

$$\frac{\Delta\sigma^2}{\sigma^2} = 2\,\frac{\Delta\sigma}{\sigma} = 2F$$

What the e-by-e average tells you that you can't learn from the INCLUSIVE average

♡ e-by-e average separates two or several classes of events with different μ and σ

♡ e.g. Two-Component Models

Consider a compound distribution with 2 components, of which only one component appears on any given event. Represent the distributions as Gamma distributions, for convenience:

$$f_c(p_T) = q\Gamma(p_T, p_1, b_1) + (1-q)\Gamma(p_T, p_2, b_2), \tag{14}$$

where q and $1-q$ are the probabilities for an event to have either component distribution. Consider 2 simple cases:

A) Same μ different σ. $\Delta T = 1/b_2 - 1/b_1$

B) Different μ same σ.

A third case related to B has a distribution represented by $T = 1/b$ which varies continuously from event to event.

B') Continuously varying T From a recent paper [R. Korus, *et al.* Phys. Rev. C**64**, 054908 (2001)] T varies with a mean $\langle T\rangle$ and standard deviation σ_T.

Note that in all cases the mean and standard deviation of the compound distribution must equal the measured inclusive values(recall Fig. 1).

♡ For B and B' the e-by-e effect is seen at the level of the moments

$$\frac{\sigma^2_{\bar{x}_T}}{\mu^2_T} = \frac{1}{n}\frac{\sigma^2_{x_T}}{\mu^2_T} + (1-\frac{1}{n})\frac{\sigma^2_T}{\langle T\rangle^2} \quad . \tag{15}$$

If you divide by random, you get an additional factor of n:

$$F = \frac{p}{2}(\langle n\rangle - 1)\frac{\sigma^2_T}{\langle T\rangle^2} \quad , \tag{16}$$

where p is the parameter of the inclusive Γ distribution.

♡ **For A the e-by-e effect cannot be seen at the level of the moments.** The standard deviation $\sigma_{\bar{x}_c}$, of the compound distribution scales as $1/n$, exactly like the simple distribution, so that the e-by-e effect can not be seen from the variance.

$$\frac{\sigma^2_{\bar{x}_c}}{\mu^2} = \frac{1}{n}\frac{\sigma^2_{x_c}}{\mu^2}$$

However the detailed shape of the e-by-e distributions are different for the simple and compound cases.

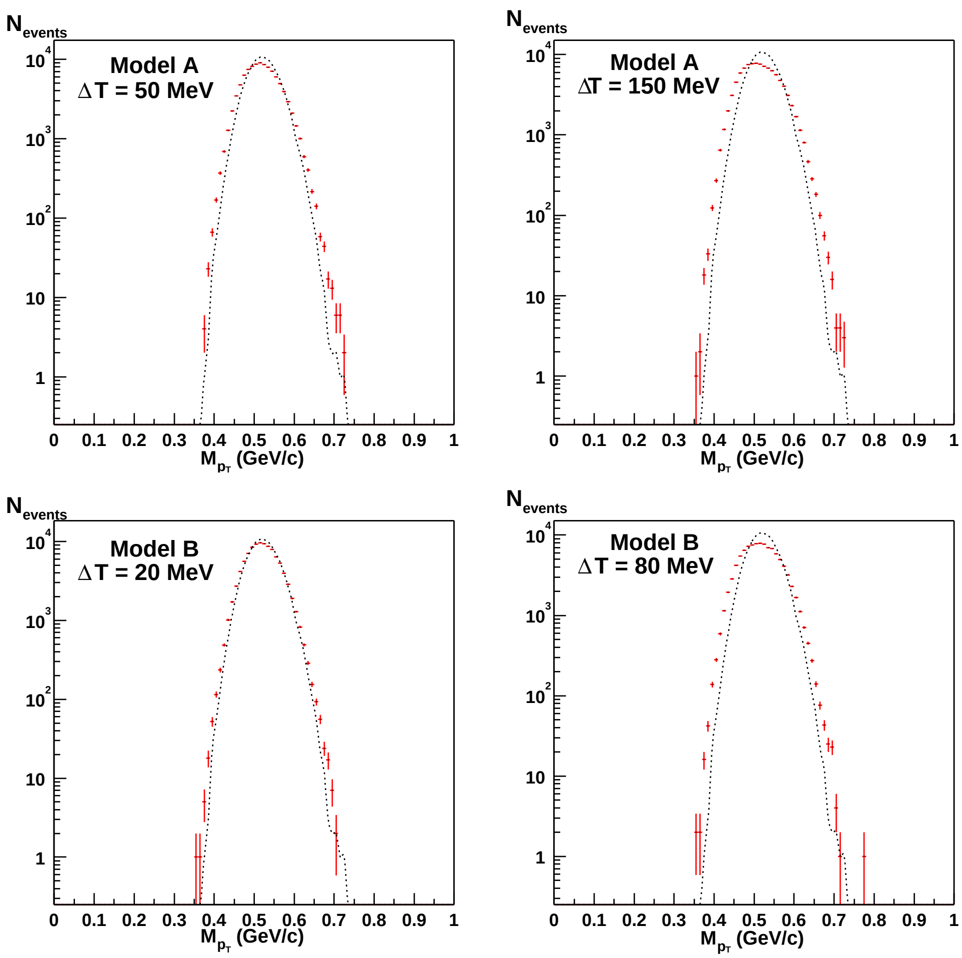

Figure 5: Comparison of the PHENIX baseline mixed event distribution for 5% centrality (Fig. 4) to the 2-component models (red points). Semi-inclusive p_T distribution for data and models shown in Fig. 1.

PHENIX limits on ΔT for 2 models

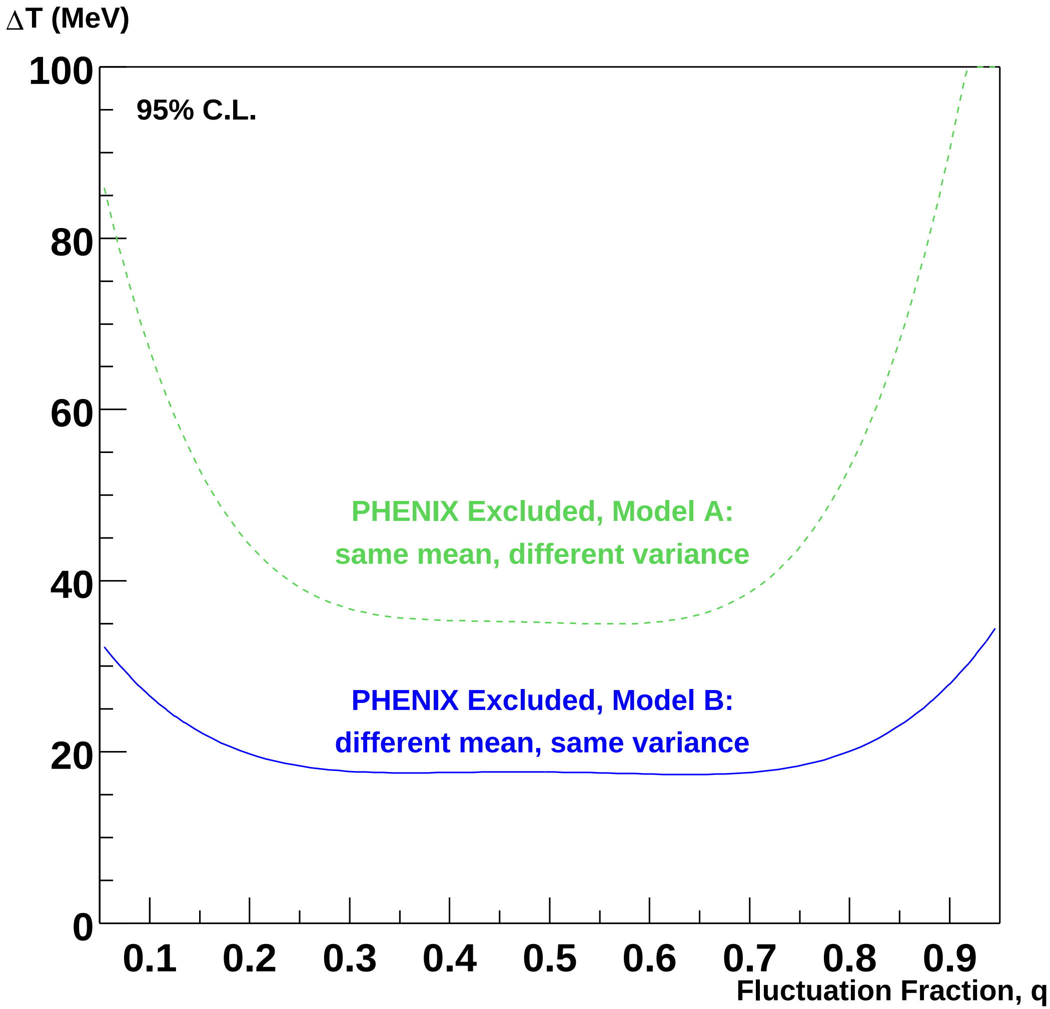

Figure 6: PHENIX 95% confidence limits excluding ΔT for 2-component distributions in cases A and B. The curves represent the lower boundaries of the excluded regions.

Centrality class	$\langle n \rangle$	σ_n	σ_{p_T} (MeV/c)	$\sigma_{M_{p_T}}$ (MeV/c)	F (%)	ϕ_{p_T} (MeV/c)
0 - 5 %	59.6	10.8	290	38.6	1.9 ± 2.1	5.65 ± 6.02
0 - 10 %	53.9	12.2	290	41.1	2.0 ± 2.5	6.03 ± 7.28
10 - 20 %	36.6	10.2	290	49.8	2.1 ± 2.2	6.11 ± 6.63
20 - 30 %	25.2	7.8	289	61.1	1.8 ± 3.0	5.47 ± 9.16

Numerical Values of PHENIX results.

Φ—The variable I love to hate.

From NA49 and Gazdzicki and Mrowczynski Z. Phys. C**54**, 127 (1992):

$$\Phi_{p_T} = F \times \sigma_{p_T}$$

In words, Φ_{p_T} multiplies the fractional difference between the e-by-e $\sigma_{M_{p_T}}$ and the random baseline by the INCLUSIVE σ_{p_T}.

Suppose the e-by-e σ is $\sim 1\%$ of the inclusive σ and you find a 1% difference in the e-by-e σ from random, why would you claim that this is equivalent to $\Phi = 1\%$ of the inclusive σ, which in this example equals the ENTIRE e-by-e σ ???

BUT

The PHENIX result illustrates exactly what Φ is supposed to be good for: It is supposed to correct out the centrality dependence. PHENIX data show $F =$ constant, independent of centrality, which is non-intuitive to me but gives Φ independent of centrality?

Interestingly STAR sees the same effect.

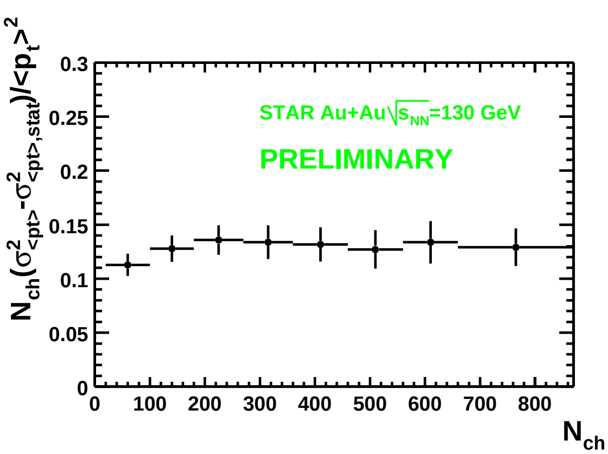

Figure 7: Star results for F vs n [nucl-ex/0109006], where I have assumed that $\sigma^2_{\langle p_T \rangle,\mathrm{stat}}/\mu^2 = 1/(np) = 1/2n$, in which case the label on the y axis is equal to F

- But, I think Star result for F is 6 times larger than PHENIX'!

Clearly, much left to be understood.

Conclusions.

• PHENIX data indicate very small, if any, non-random effect. Similar to NA49 at mid-rapidity at SPS.

• Both PHENIX and NA49 central M_{p_T} distributions agree with Gamma distributions calculated assuming statistically independent particle emission from the semi-inclusive p_T distribution.

• PHENIX and STAR limits/measurements for non-random fraction of $\sigma_{M_{p_T}}$ are both independent of centrality

BUT

♠ Appear to disagree by a factor of 6 where they overlap.

• Continue looking for smaller fluctuations— Especially the known Bose-Einstein correlation/fluctuation, to prove the sensitivity, and hopefully to discover something.