Gauge Links and TMD-Factorization

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Status overview of transverse momentum dependent factorization theorems with an emphasis on evolution, universality/non-universality, and the issue of factorization breaking:

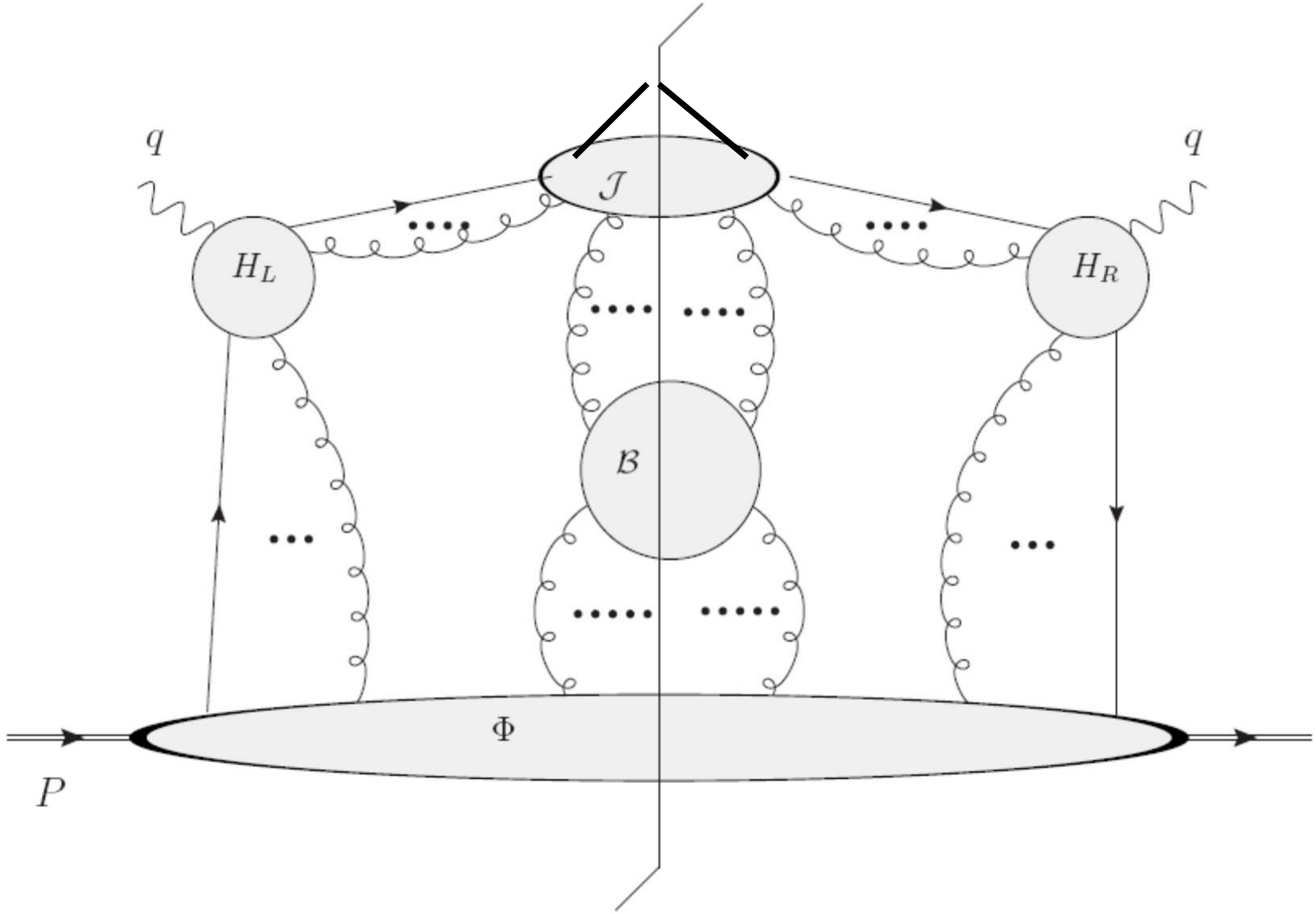
- Review basic concepts and complications with definitions.
- Factorization Breaking.
- Combining existing fits into evolved, momentum space TMDs.
- Future directions.

BNL Workshop on Drell-Yan Physics, May 11, 2011

TMD-Factorization:

- Approximations

SIDIS



TMD-Factorization:

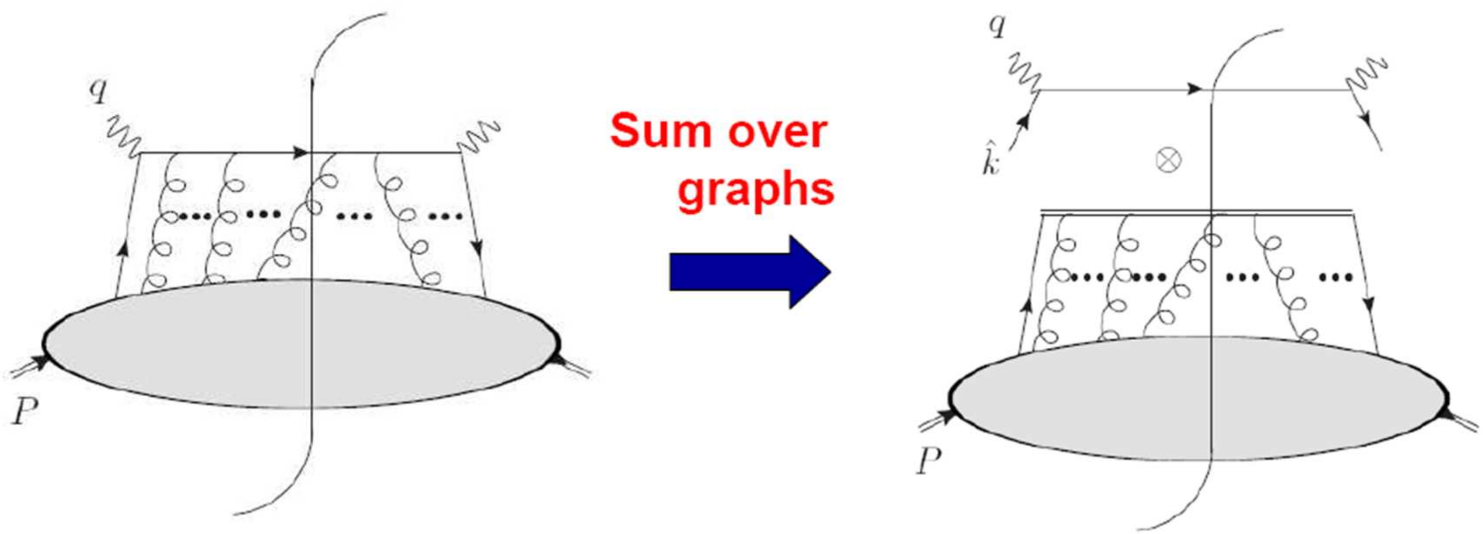
- Complications with defining TMDs:

Gauge Links/Wilson Lines

- Integrated PDF:

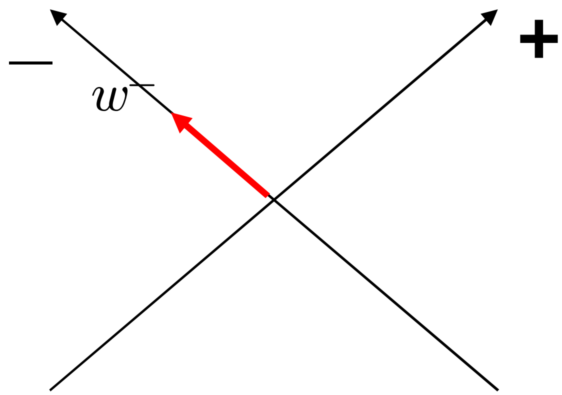
$$f(x) = \text{F.T.} \, \langle p | \, \bar{\psi}(0, w^-, \mathbf{0}_t) \underline{V_w^\dagger(u_\mathrm{J})} \gamma^+ \underline{V_0(u_\mathrm{J})} \psi(0) | p \rangle$$

$$u_\mathrm{J} = (0, 1, \mathbf{0}_t)$$



Gauge Links/Wilson Lines

- Paths of Wilson lines in coordinate space:



Standard (Integrated)

Gauge Links/Wilson Lines

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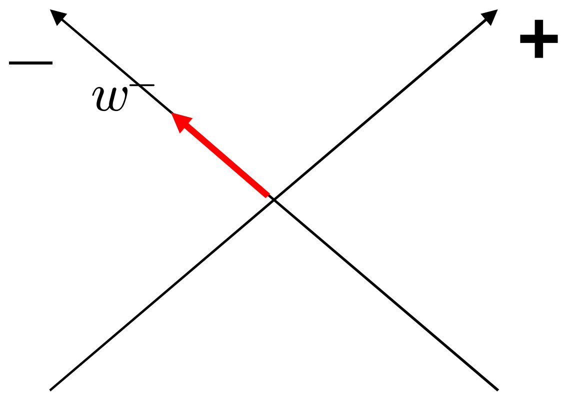
$$u_\mathrm{J} = (0, 1, \mathbf{0}_t)$$

- TMD PDF:

$$\Phi(x, \mathbf{k}_t) = \quad \text{F.T.}\, \langle p|\, \bar{\psi}(0, w^-, \mathbf{w}_t) \quad \underline{??} \quad \psi(0)\, |p\rangle$$

TMD PDFs: Gauge Links/Wilson Lines

- Paths of Wilson lines in coordinate space:



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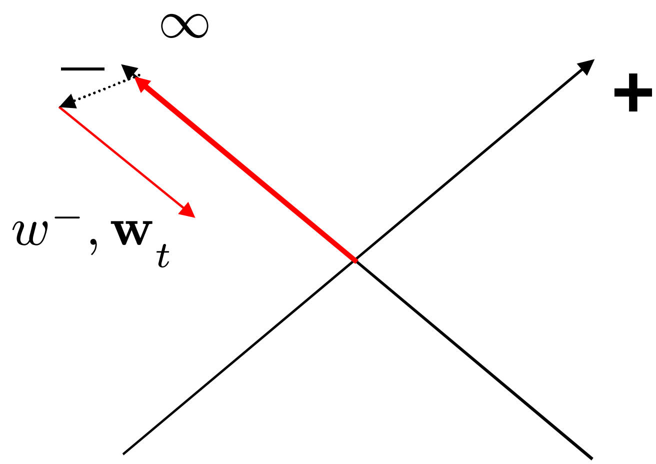

Unintegrated First Try

TMD-Factorization:

- Complications with defining TMDs:
  - Divergences.
  - Wilson lines / gauge links.
  - Universality vs. non-universality.
  - ***Definitions dictated by requirements for factorization!***

TMD PDFs: Gauge Links/Wilson Lines

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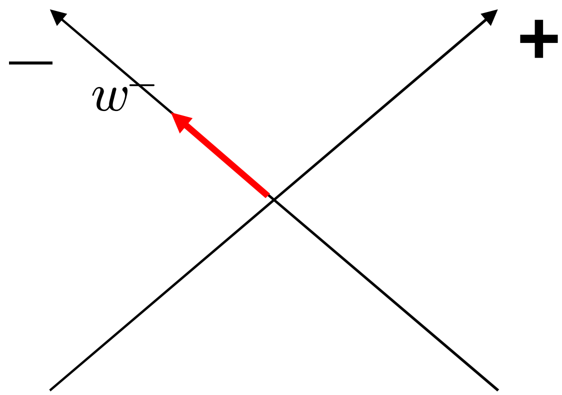

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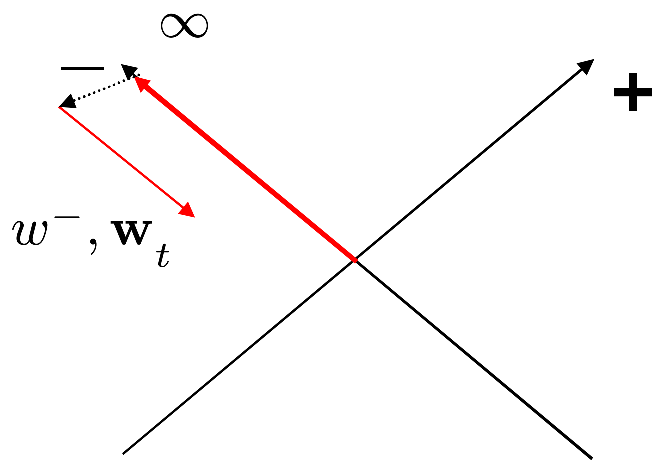

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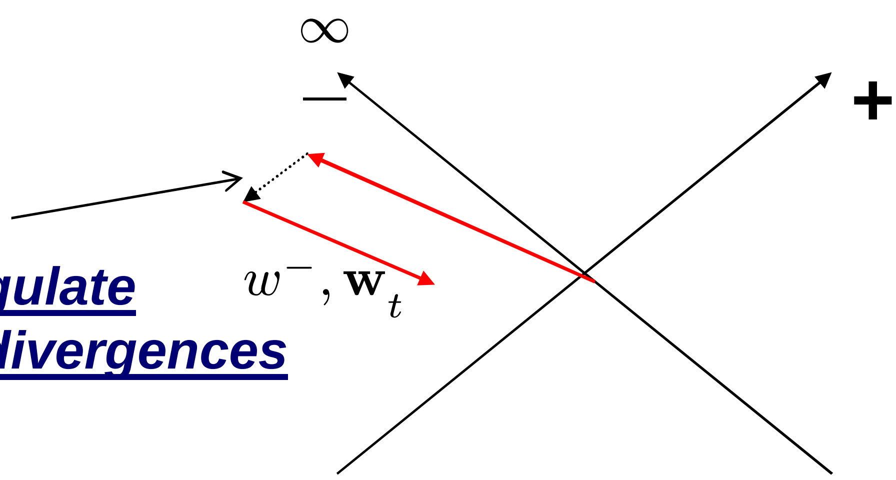
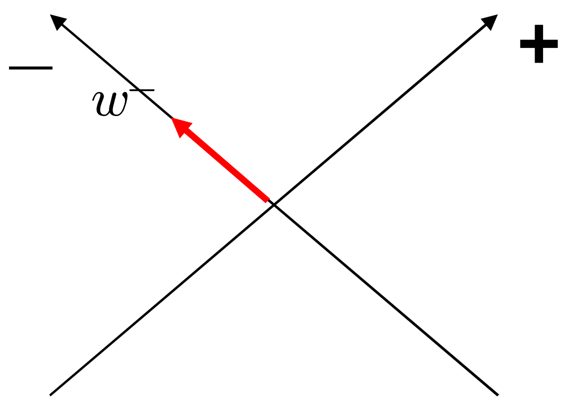


Tilt to regulate rapidity divergences

Unintegrated "tilted" Wilson lines

TMD PDFs: Gauge Links/Wilson Lines

- Paths of Wilson lines in coordinate space:



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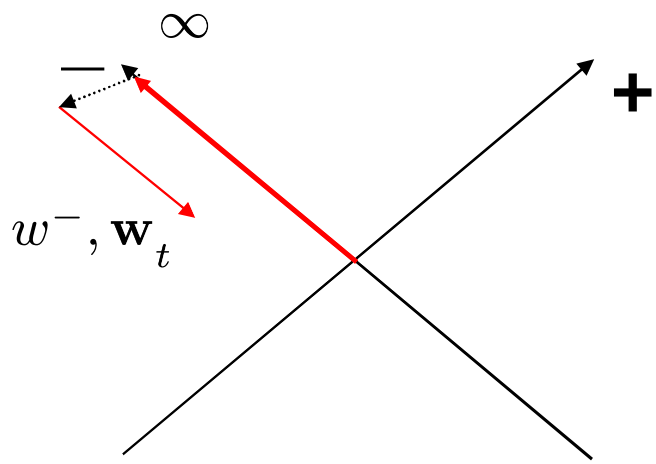

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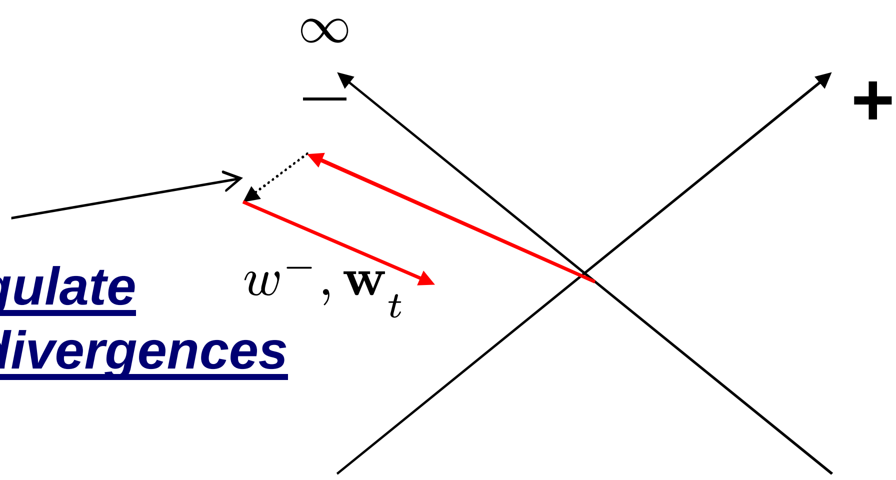

Tilt to regulate rapidity divergences

Still more complications!

Unintegrated "tilted" Wilson lines

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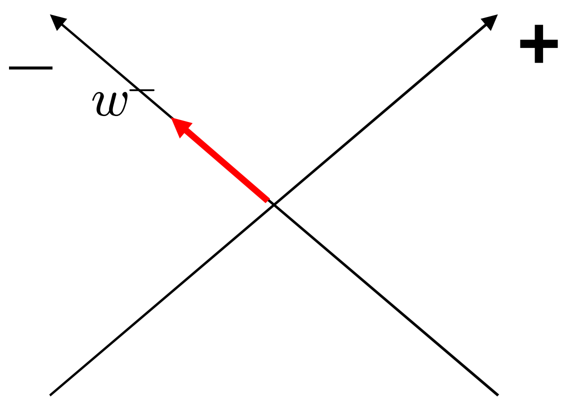

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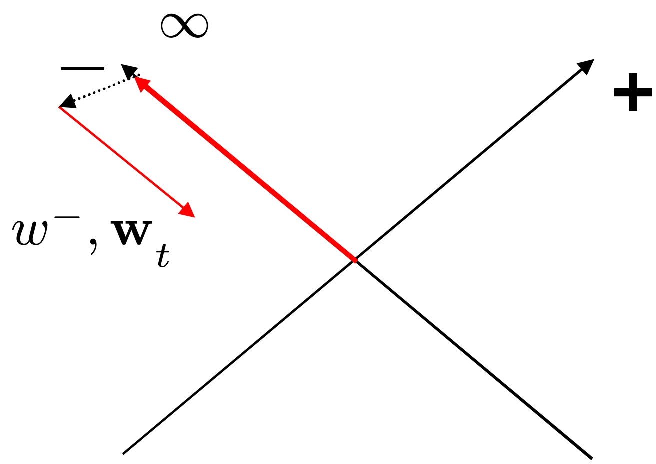

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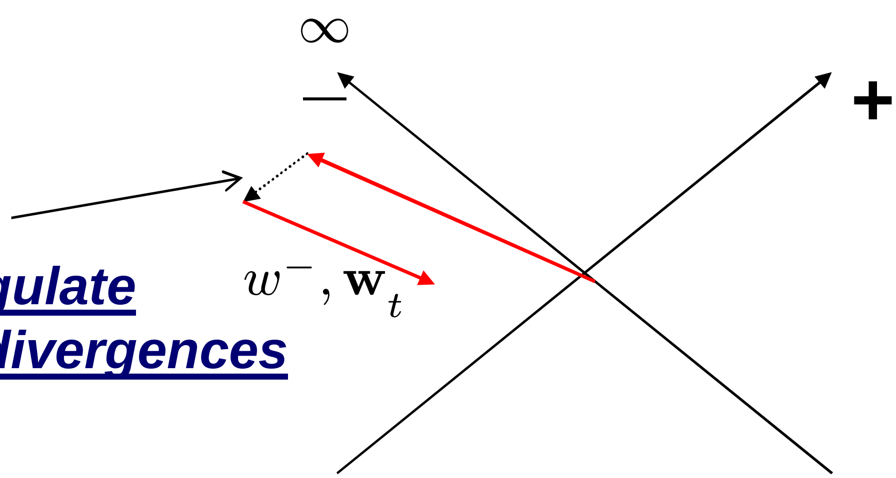


Tilt to regulate rapidity divergences

Unintegrated "tilted" Wilson lines

Still more complications!

More on definitions to come!

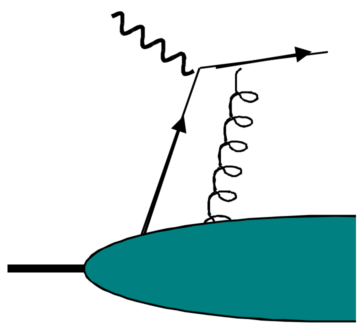
TMD-Factorization:

- Complications with defining TMDs:
  - Divergences.
  - Wilson lines / gauge links.
  - Universality vs. non-universality.
  - ***Definitions dictated by requirements for factorization!***

- Processes:
  - Semi-Inclusive deep inelastic scattering.
  - Drell-Yan.
  - e^+/e^- annihilation.
  - $p + p \Longrightarrow h_1 + h_2 + X$

Universality

- Direction of "gauge link" in the TMD definition matters!



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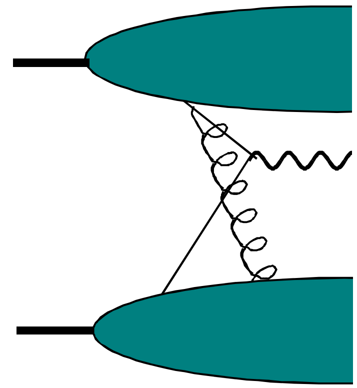

Drell-Yan

- Direction of gauge links matters.
  - Sivers function - sign-flip.

(Brodsky, Hwang, Schmidt (2002); Collins (2002))

TMD-Factorization:

- Complications with defining TMDs:
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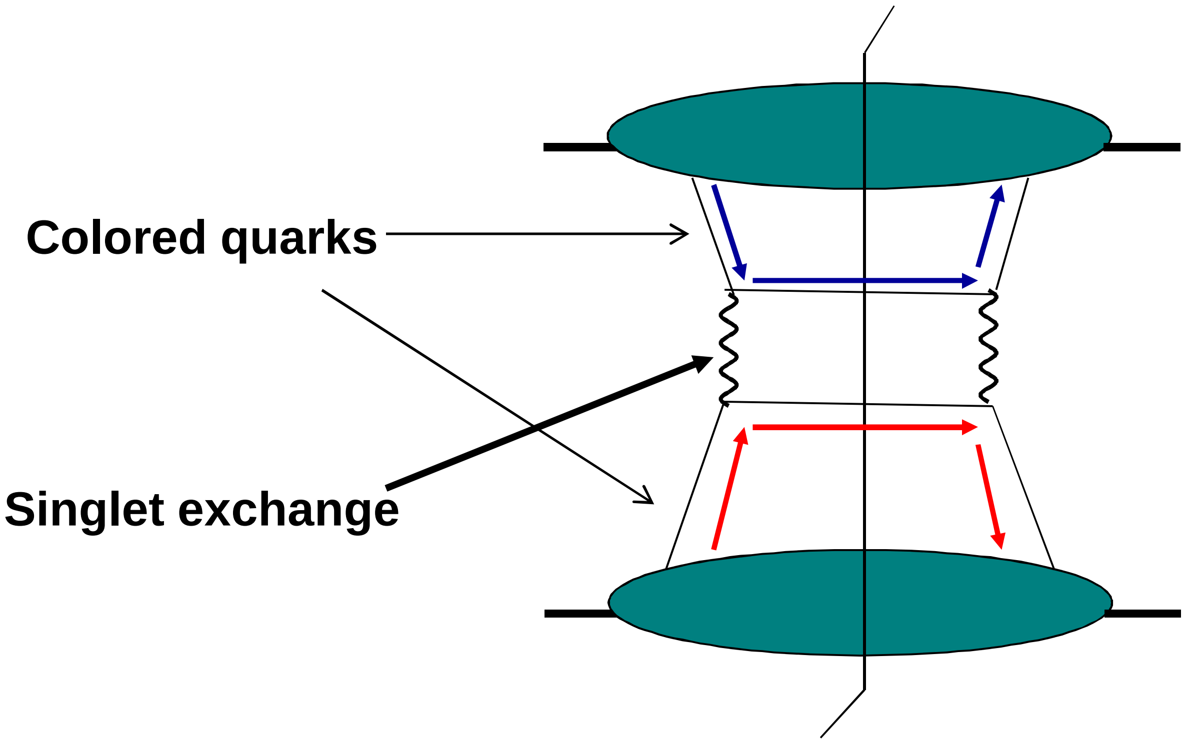
Watch out for sign flips!

Universality

- Hadro-production of hadrons:



Universality

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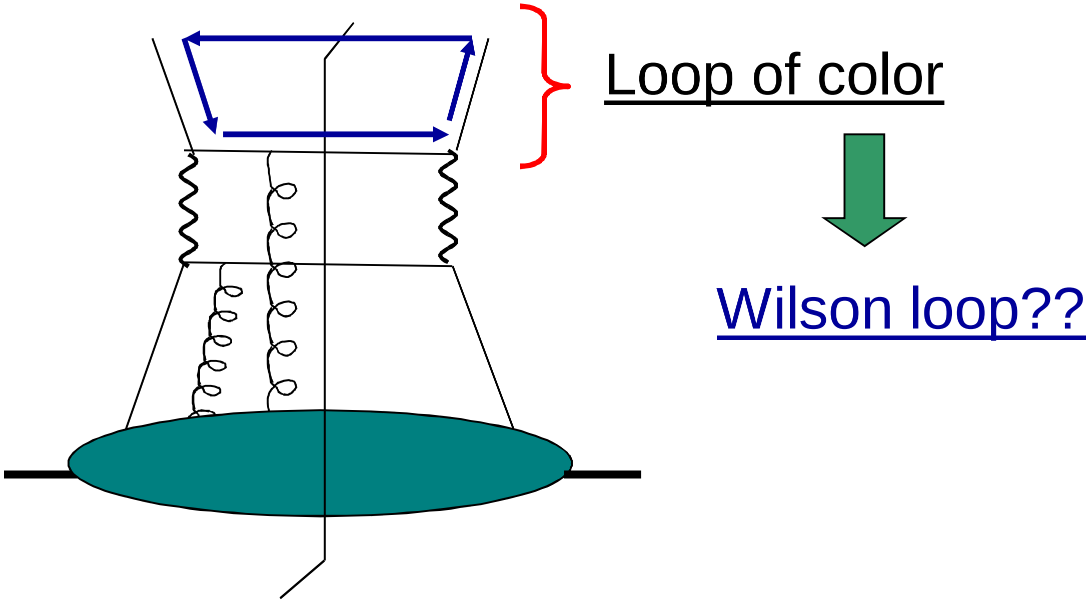


- *G*auge link is *at least* non-standard.

(Bomhof, Mulders, Pijlman (2004);
Collins, Qiu (2007);
Collins (2007))

“Generalized” TMD-Factorization

- “Generalized” TMD-Factorization: Formal factorization formula exists, *but parton distributions are non-universal!*

(!!?)

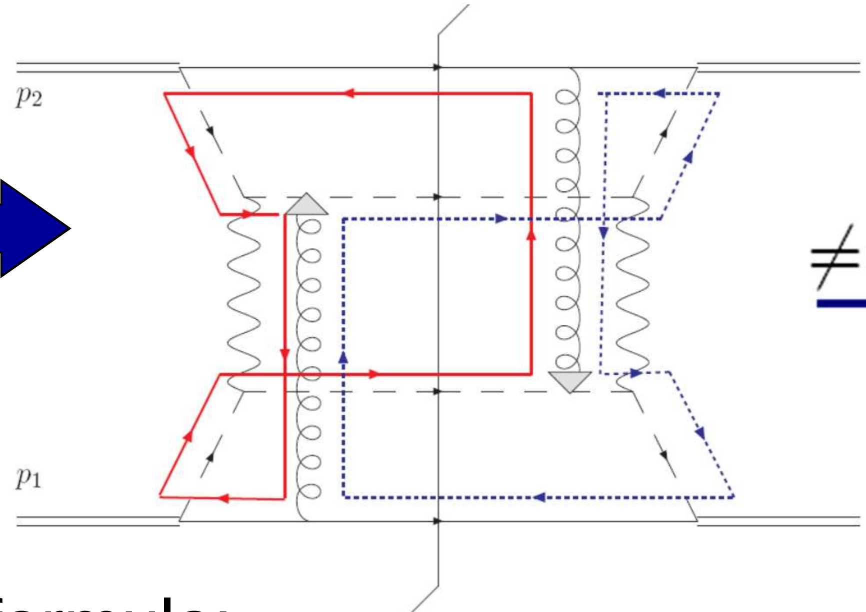
$$d\sigma \sim \mathcal{H} \otimes \Phi_{P_1}^{[+(\Box)]}(x_1, k_{1T}) \otimes \Phi_{P_2}^{[+(\Box)]}(x_2, k_{2T}) \otimes \cdots$$

Generalized TMD-factorization breaking:

- Gluons have color.

Actual color structure

Color Entanglement

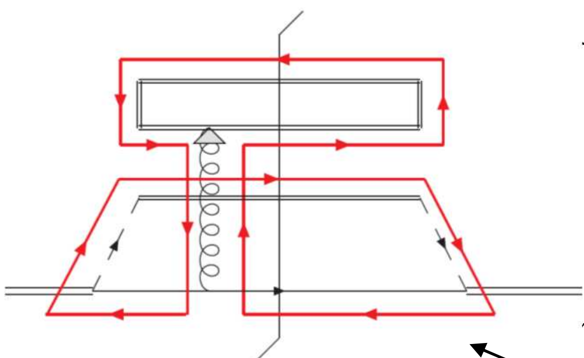


$\neq 0!$

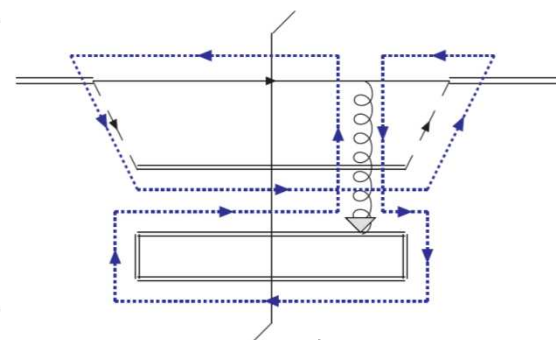
- "generalized" factorization formula:

(TCR, Mulders (2010))

$\mathcal{H}$ x



x



$= 0$

$\mathrm{Tr}_C\,[t^a] = 0$

TMD-Factorization:

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- Processes:
  - Semi-Inclusive deep inelastic scattering. ✓
  - Drell-Yan. ✓
  - e^+/e^- annihilation. ✓
  - ~~$p + p \rightarrow h_1 + h_2 + X$~~ **!!**

Watch out for sign flips!

TMD PDF is not just non-universal, it is ill-defined at the operator level!

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- Implementation and TMD phenomenology.

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- Implementation and TMD phenomenology.
  - Use existing fixed-scale fits / no evolution.

TMD-Factorization

- TMD Parton model intuition (Drell-Yan):

$$W^{\mu\nu} = \sum_f \left|\mathcal{H}_f(Q)^2\right|^{\mu\nu} \int d^2\mathbf{k}_{1T}\, d^2\mathbf{k}_{2T}\, F_{f/P_2}(x_1, \mathbf{k}_{1T})\, F_{\bar{f}/P_2}(x_2, \mathbf{k}_{2T}) \times$$
$$\times\, \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{k}_{2T} - \mathbf{q}_T)$$

Leading order hard part — $\left|\mathcal{H}_f(Q)^2\right|^{\mu\nu}$

No evolution — $F_{f/P_2}(x_1, \mathbf{k}_{1T})$, $F_{\bar{f}/P_2}(x_2, \mathbf{k}_{2T})$

Generalized Parton Model

TMD-Factorization:

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  } **Watch out for sign flips!**

- Implementation and TMD phenomenology.
  - Use existing fixed-scale fits / no evolution.
  - Use existing "old fashion" implementation of Collins-Soper-Sterman formalism.

Evolved Cross Section:

- Typical appearance of Collins-Soper-Sterman implementation:

(Contrast with GPM picture.)

$$d\sigma \sim \int d^2\mathbf{b}\, e^{-i\mathbf{b}\cdot\mathbf{q}_T}$$

$$\times \int_{x_1}^{1} \frac{d\hat{x}_1}{\hat{x}_1} \tilde{C}_{f/j}(x_1/\hat{x}_1, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P_1}(\hat{x}_1, \mu_b)$$

$$\times \int_{x_2}^{1} \frac{d\hat{x}_2}{\hat{x}_2} \tilde{C}_{f/j}(x_2/\hat{x}_2, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P_2}(\hat{x}_2, \mu_b)$$

$$\times \exp\left[\int_{1/b^2}^{Q^2} \frac{d\mu'^2}{\mu'^2} \left\{\mathcal{A}(\alpha_s(\mu')) \ln\frac{Q^2}{\mu'^2} + \mathcal{B}(\alpha_s(\mu'))\right\}\right]$$

$$\times \exp\left[-g_k(b) \ln\frac{Q^2}{Q_0^2} - g_1(x_1, b) - g_2(x_2, b)\right]$$

+ Large q_T term

Definition given in CSS derivation, but hard to indentify what should appear in tables.

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Watch out for sign flips!

- Implementation and TMD phenomenology.
  - Use existing fixed-scale fits / no evolution.
  - Use existing "old fashion" implementation of Collins-Soper-Sterman formalism.
  - ***Full TMD formalism, including evolution.***

(New Collins Definitions)

What is needed?

- TMD Parton model intuition (Drell-Yan):

$$W^{\mu\nu} = \sum_f \left|\mathcal{H}_f(Q)^2\right|^{\mu\nu} \int d^2\mathbf{k}_{1T}\, d^2\mathbf{k}_{2T}\, F_{f/P_2}(x_1, \mathbf{k}_{1T})\, F_{\bar{f}/P_2}(x_2, \mathbf{k}_{2T}) \times$$

$$\times\, \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{k}_{2T} - \mathbf{q}_T)$$

- Using newest definitions:

$$W^{\mu\nu} = \sum_f |\mathcal{H}_f(Q;\mu)^2|^{\mu\nu}$$

$$\times \int d^2\mathbf{k}_{1T}\, d^2\mathbf{k}_{2T}\, F_{f/P_1}(x_1, \mathbf{k}_{1T}; \mu; \zeta_1)\, F_{\bar{f}/P_2}(x_2, \mathbf{k}_{2T}; \mu; \zeta_2)$$

$$\times\, \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{k}_{2T} - \mathbf{q}_T)$$

$$+\, Y(Q, q_T) + \mathcal{O}((\Lambda/Q)^a).$$

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Process dependence in hard part

Universal PDFs with evolution

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$$\times\, \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{k}_{2T} - \mathbf{q}_T)$$
$$+\, Y(Q, q_T) + \mathcal{O}((\Lambda/Q)^a).$$

Large Transverse Momentum Correction

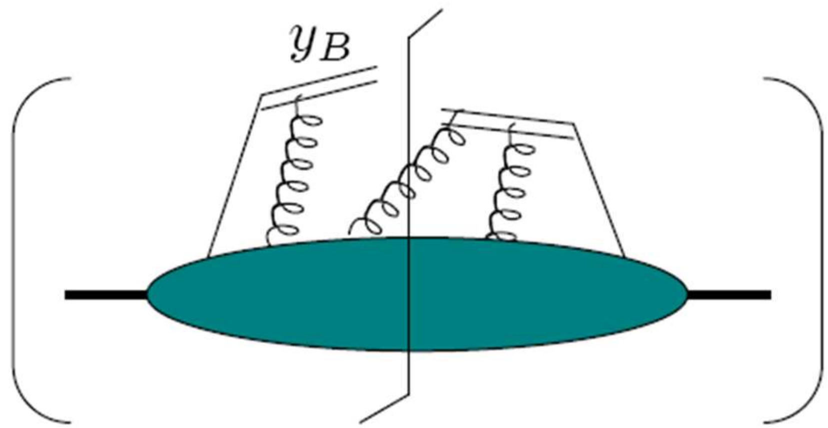
Process dependence in hard part

Universal PDFs with evolution

More on Definitions

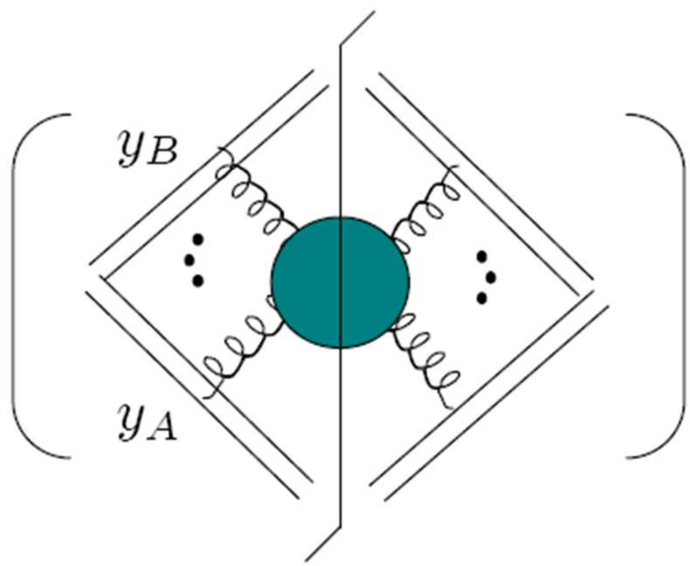
- "Unsubtracted" TMD PDF:

$$\tilde{F}^{\text{unsub}}_{f/P}(x, \mathbf{b}; \mu; y_P - y_B) \sim \text{F.T.}$$



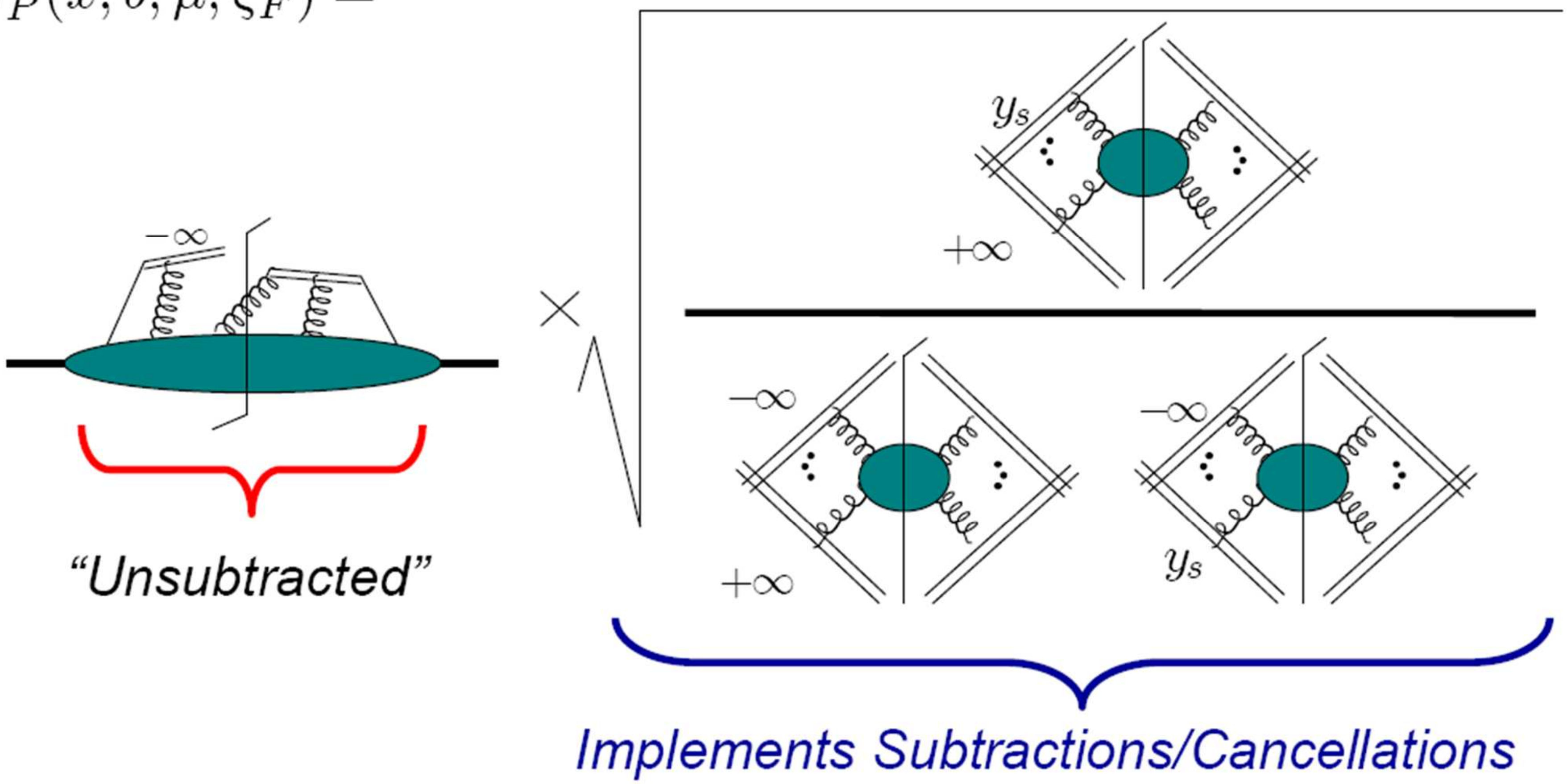
- Soft Factor:

$$\tilde{S}(\mathbf{b}; y_A, y_B) \sim \text{F.T.}$$



TMD PDF, Complete Definition:

$F_{f/P}(x, b; \mu; \zeta_F) =$



From *Foundations of Perturbative QCD*, J.C. Collins,
(See also, Collins, TMD 2010 Trento Workshop)

Current Strategy:

- Use evolution to combine existing fits into unified/global fits that include evolution.

  (S.M. Aybat, TCR (2011))

  – PDFs:
    - Start with DY:

      (Landry et al, (2003); Konychev, Nadolsky (2006)) (BLNY)
    - Modify to match to SIDIS:

      (Schweitzer, Teckentrup, Metz (2010)) (STM)

- Can supply explicit, evolved TMD PDF fit.

Evolution

- Collins-Soper Equation:
  - $$-\frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu)$$

Perturbatively calculable from definition at small b.

$$\tilde{K}(b_T; \mu) = \frac{1}{2} \frac{\partial}{\partial y_n} \ln \frac{\tilde{S}(b_T; y_n, -\infty)}{\tilde{S}(b_T; +\infty, y_n)}$$

- RG:
  - $$\frac{d\tilde{K}}{d \ln \mu} = -\gamma_K(g(\mu))$$
  - $$\frac{d \ln \tilde{F}(x, b_T; \mu, \zeta)}{d \ln \mu} = -\gamma_F(g(\mu); \zeta/\mu^2)$$

Perturbatively calculable, from definitions

Implementing Evolution

- After evolution:

$$\tilde{F}_{f/H}(x, b_T, \mu, \zeta) = \sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_b, g(\mu_b)) f_{j/H}(x, \mu_b) \times \quad \text{A}$$

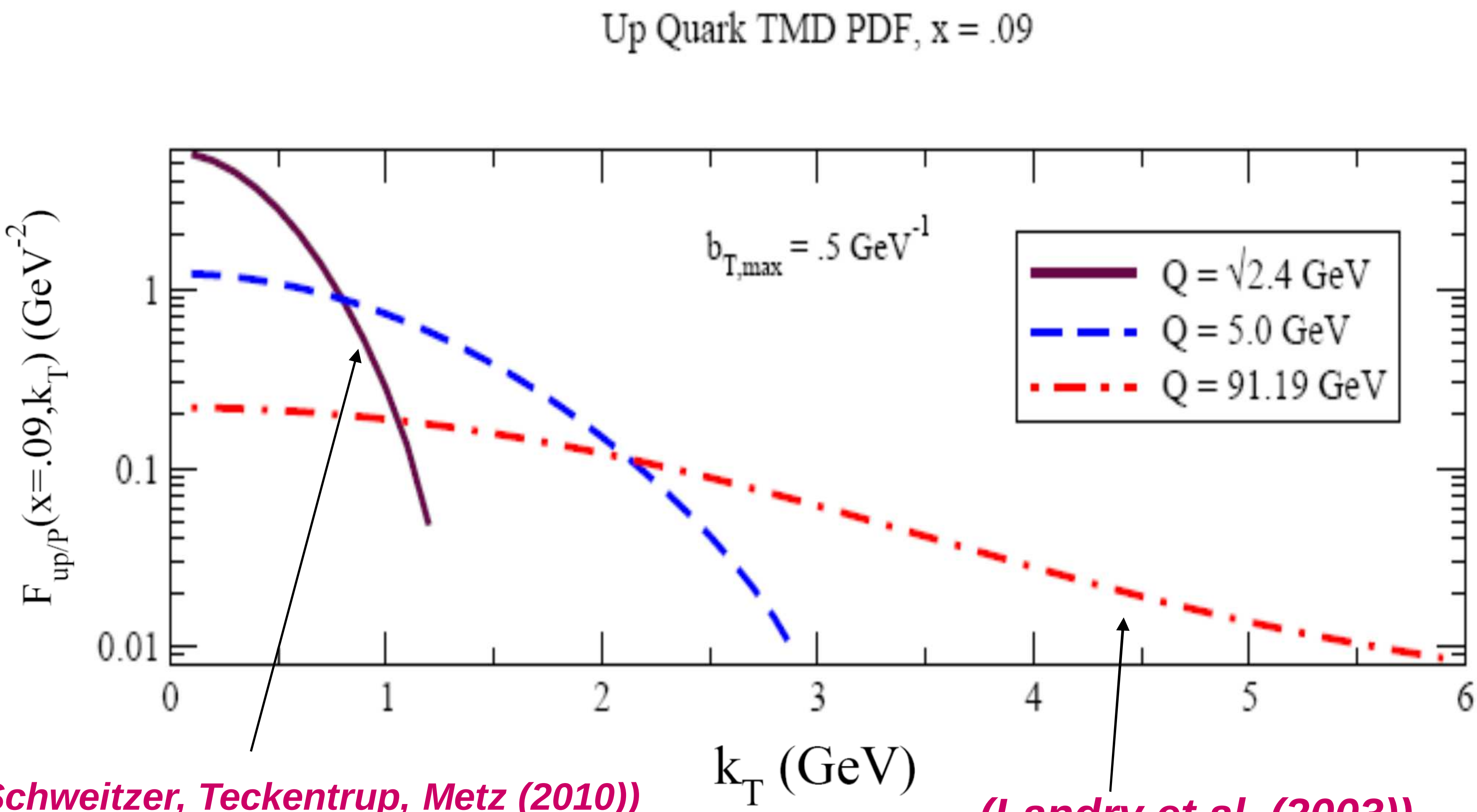
$$\times \exp\left\{ \ln \frac{\sqrt{\zeta}}{\mu_b} \tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln \frac{\sqrt{\zeta}}{\mu'} \gamma_K(g(\mu')) \right] \right\} \times \quad \text{B}$$

$$\times \exp\left\{ g_{j/H}(x, b_T) + g_K(b_T) \ln \frac{\sqrt{\zeta}}{Q_0} \right\} \quad \text{C}$$

$$b_*(b_T) \equiv \frac{b_T}{\sqrt{1 + b_T^2/b_{max}^2}} \qquad \mu_b(b_T) \sim 1/b_*$$

CSS matching procedure

Evolving TMD PDFs



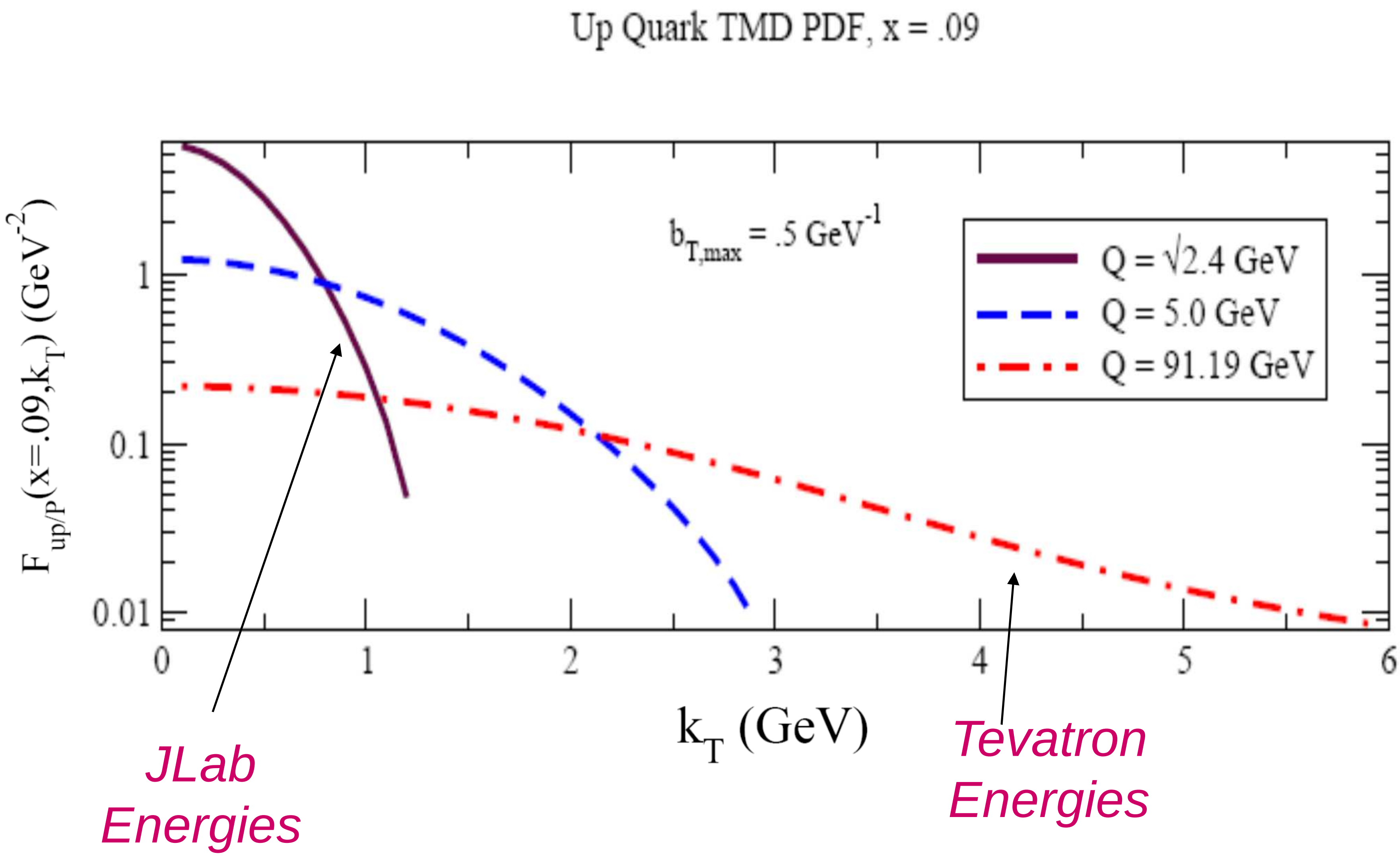
(Schweitzer, Teckentrup, Metz (2010))

(SIDIS)

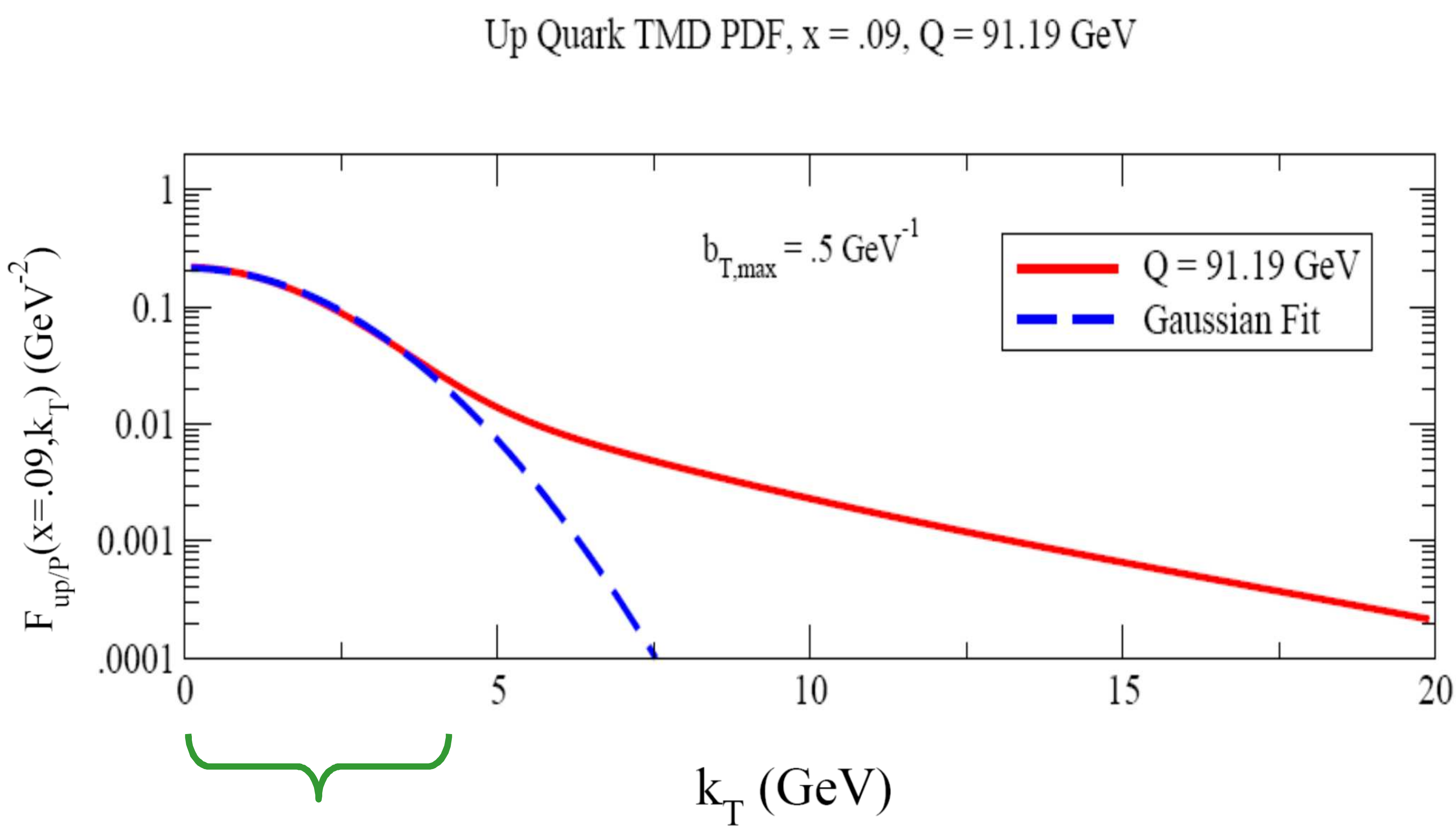
(Landry et al, (2003))

(Drell-Yan)

Evolving TMD PDFs



Evolving TMD PDFs



Gaussian fit good at small k_T.

Unambiguous Hard Part

- Higher orders follow systematically from definitions:

$$W^{\mu\nu} = |\mathcal{H}_f(Q;\mu/Q)^2|^{\mu\nu}\; F_{f/P_1} \otimes F_{f/P_2}$$

$$\Longrightarrow \quad |\mathcal{H}_f(Q;\mu/Q)^2|^{\mu\nu} = \frac{W^{\mu\nu}}{F_{f/P_1} \otimes F_{f/P_2}}$$

Unambiguous Hard Part

- Definition:

$$|\mathcal{H}_f(Q;\mu/Q)^2|^{\mu\nu} = \frac{W^{\mu\nu}}{F_{f/P_1} \otimes F_{f/P_2}}$$

- Drell-Yan:

Unambiguous Hard Part

- Definition:

$$|\mathcal{H}_f(Q;\mu/Q)^2|^{\mu\nu} = \frac{W^{\mu\nu}}{F_{f/P_1} \otimes F_{f/P_2}}$$

- Drell-Yan: $(\overline{\mathrm{MS}})$

$$|\mathcal{H}_f(Q;\mu/Q)^2|^{\mu\nu} =$$

$$e_f^2 |H_0^2|^{\mu\nu} \left(1 + \frac{C_F \alpha_s}{\pi} \left[\frac{3}{2} \ln\left(Q^2/\mu^2\right) - \frac{1}{2} \ln^2\left(Q^2/\mu^2\right) - 4 + \frac{\pi^2}{2}\right]\right) + \mathcal{O}(\alpha_s^2)$$

Unambiguous Hard Part

- Definition:

$$|\mathcal{H}_f(Q;\mu/Q)^2|^{\mu\nu} = \frac{W^{\mu\nu}}{F_{f/P_1} \otimes F_{f/P_2}}$$

- Drell-Yan: $(\overline{\text{MS}})$

$$|\mathcal{H}_f(Q;\mu/Q)^2|^{\mu\nu} =$$
$$e_f^2|H_0^2|^{\mu\nu}\left(1+\frac{C_F\alpha_s}{\pi}\left[\frac{3}{2}\ln\left(Q^2/\mu^2\right)-\frac{1}{2}\ln^2\left(Q^2/\mu^2\right)-4+\frac{\pi^2}{2}\right]\right)+\mathcal{O}(\alpha_s^2)$$

- SIDIS

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Space-like photon!

$$|\mathcal{H}_f(Q;\mu/Q)^2|^{\mu\nu} =$$
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Long-Term Goal:

- Repository of improved TMD fits with evolution.

https://projects.hepforge.org/tmd/

- Based on well-understood operator definitions.

(Collins Definitions)

Agenda:

- Extend to higher orders, calculate all Y-terms.
  Large/small b_T matching.
  Improved global fits including DY, SIDIS e+e- annihilation…
  (In progress…)

- Extend to polarization dependent functions (Sivers, Boer-Mulders, etc…). *(In progress…)*

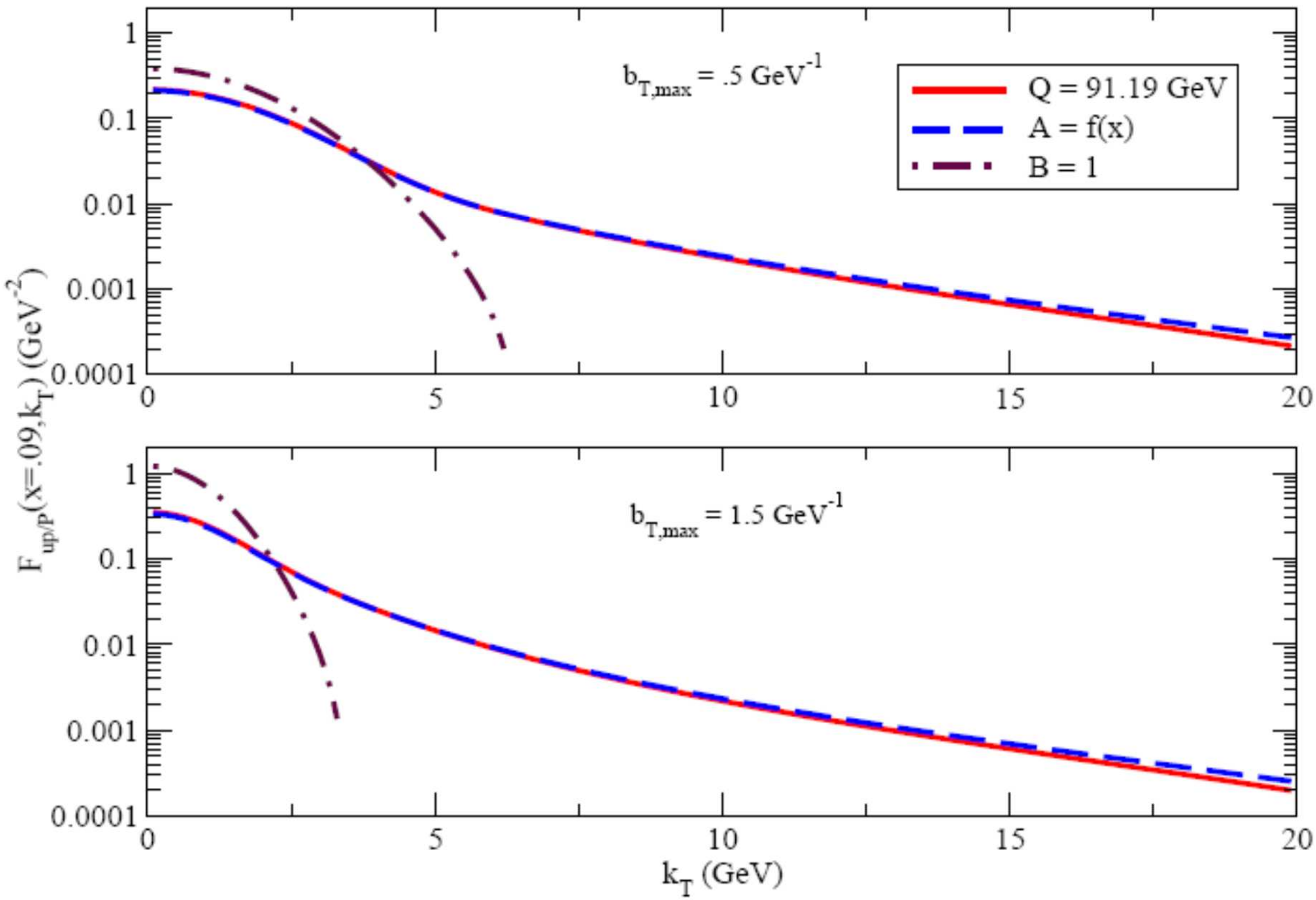
- TMD gluon distribution.
  (Higgs…)

- Factorization breaking??

Thanks!

Backup Slides

Up Quark TMD PDF, x = .09, Q = 91.19 GeV



Understanding the Definition:

- Start with only the hard part factorized:

Naïve Factorization:

$$d\sigma = |\mathcal{H}|^2 \, \tilde{F}_1^{\mathrm{unsub}}(y_1 - (-\infty)) \times \tilde{F}_2^{\mathrm{unsub}}(+\infty - y_2).$$

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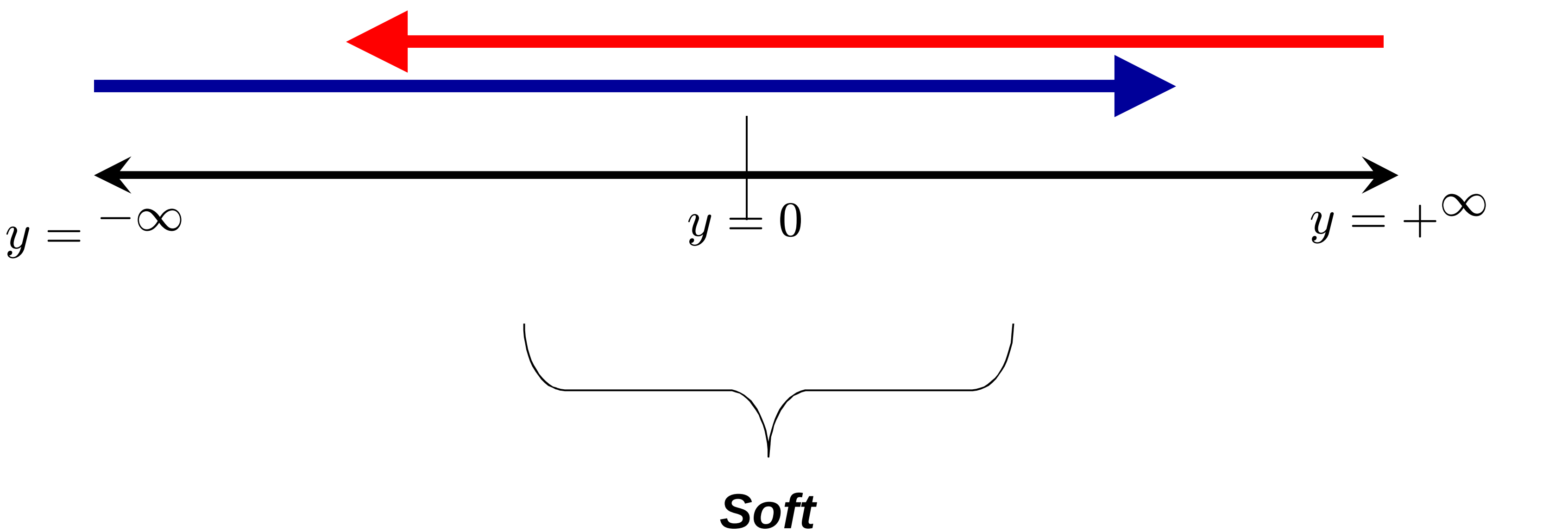

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- Separate soft part:

$$d\sigma = |\mathcal{H}|^2 \, \frac{F_1^{\rm unsub}(y_1 - (-\infty))}{\sqrt{\tilde{S}(+\infty, -\infty)}} \times \frac{\tilde{F}_2^{\rm unsub}(+\infty - y_2)}{\sqrt{\tilde{S}(+\infty, -\infty)}}.$$

- Multiply by: $\dfrac{\sqrt{\tilde{S}(+\infty, y_s)\, \tilde{S}(y_s, -\infty)}}{\sqrt{\tilde{S}(+\infty, y_s)\, \tilde{S}(y_s, -\infty)}}$

- Rearrange factors: $d\sigma = |\mathcal{H}|^2 \, \left\{ F_1^{\rm unsub}(y_1 - (-\infty)) \sqrt{\dfrac{\tilde{S}(+\infty, y_s)}{\tilde{S}(+\infty, -\infty)\tilde{S}(y_s, -\infty)}} \right\}$

$$\times \left\{ \tilde{F}_2^{\rm unsub}(+\infty - y_2) \sqrt{\frac{\tilde{S}(y_s, -\infty)}{\tilde{S}(+\infty, -\infty)\tilde{S}(+\infty, y_s)}} \right\}$$

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Separately Well-defined