Introduction: Collinear and TMD Factorization for Drell-Yan Production

Workshop on Opportunities for Drell-Yan at RHIC

May 6, 2011

George Sterman, Stony Brook

Some general considerations to set the stage.
Apply to spin averaged and dependent.

1. Drell-Yan production in the Parton Model
2. The Physical Basis of Factorization
3. Collinear Factorization at Fixed Q_T
4. TMD Factorization for Drell Yan Production
5. A Few Concluding Comments

I. Drell-Yan Production in the Parton Model

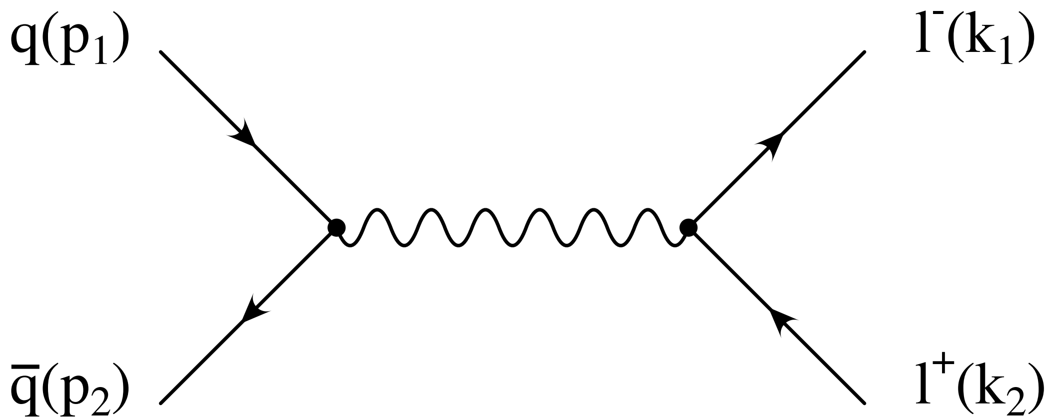
- **The original 'collinear factorization'**
- **In the parton model (1970).**
  Drell and Yan: look for the annihilation of quark pairs into virtual photons of mass Q . . . any electroweak boson in NN scattering.

$$\frac{d\sigma_{NN\to\mu\bar{\mu}+X}(Q,p_1,p_2)}{dQ^2 d\ldots} \sim$$

$$\int d\xi_1 d\xi_2 \sum_{a=\mathrm{q\bar{q}}} \frac{d\sigma^{\mathrm{EW,\,Born}}_{\mathrm{a\bar{a}}\to\mu\bar{\mu}}(Q,\xi_1 p_1,\xi_2 p_2)}{dQ^2 d\ldots}$$
$$\times(\text{probability to find parton a}(\xi_1)\text{ in }N)$$
$$\times(\text{probability to find parton }\bar{\mathrm{a}}(\xi_2)\text{ in }N)$$

The probabilities are $\phi_{q/N}(\xi_i)$'s from DIS

$\sigma^{\mathrm{EW,Born}}$ is all from this diagram (ξ's set to unity):



With this matrix element:

$$M = e_q \frac{e^2}{Q^2} \overline{u}(k_1)\gamma_\mu v(k_2)\overline{v}(p_2)\gamma^\mu u(p_1)$$

- **First square and sum/average M. Then evaluate phase space.**

- Total cross section at pair mass Q

$$\sigma^{\mathrm{EW,\,Born}}_{\mathrm{q\bar{q}}\to\mu\bar{\mu}}(x_1p_1, x_2p_2) = \frac{1}{2\hat{s}}\int\frac{d\Omega}{32\pi^2}\frac{e_q^2e^4}{3}(1+\cos^2\theta)$$
$$= \frac{4\pi\alpha^2}{9Q^2}\sum_q e_q^2$$

- With Q the pair mass and 3 for color average.

- All this is spin-averaged too, but doesn't need to be.
- Template for all hard hadron-hadron scattering
- Toward the quantum field theory of all this . . .

.

2. The Physical Basis of Factorization

- 'Collinear factorization' for hadron-hadron scattering for a hard, inclusive process with momentum transfer M to produce final state $F + X$:

$$d\sigma_{\mathrm{H_1H_2}}(p_1, p_2, M) = \sum_{a,b} \int_0^1 d\xi_a\, d\xi_b d\hat{\sigma}_{ab \to F+X}\,(\xi_a p_1, \xi_b p_2, M, \mu) \times \phi_{a/H_1}(\xi_a, \mu)\, \phi_{b/H_2}(\xi_b, \mu)$$

- Factorization proofs: justifying the "universality" of the parton distributions.

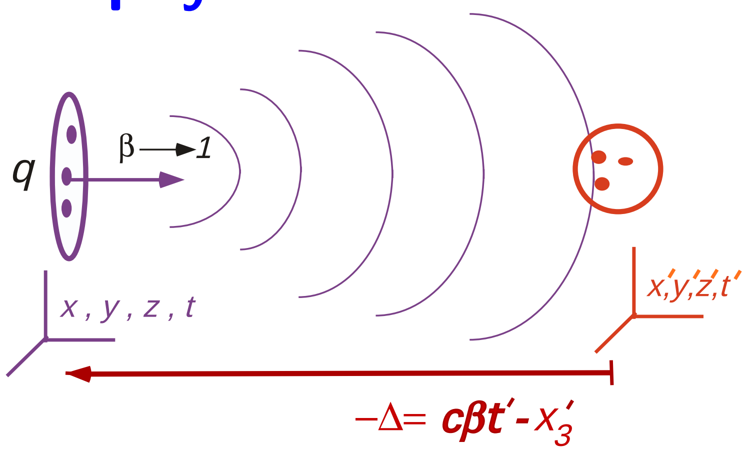
- **The operator definition:**

$$\phi_{f/H}(x,\mu,\epsilon) = \frac{1}{4N_C}\int \frac{d\lambda}{2\pi}\, \mathrm{e}^{-i\lambda x p^+}\, \langle H(p)|\bar{q}_f(\lambda u)\gamma^-\, q_f(0)|H(p)\rangle$$

- **Where the quarks are linked by ordered exponentials,**

$$\Phi_\beta^{(f)}(\lambda,0) = P\exp\left[-ig\int_0^\lambda d\eta\ \beta\cdot A^{(f)}(\eta\beta)\right] .$$

- **The physical basis: classical fields**



$$\Delta \equiv x'_3 - \beta c t'$$

- **Why a classical picture isn't far-fetched . . .**

The correspondence principle is the key to IR divergences.

An accelerated charge must produce classical radiation,

and an infinite numbers of soft gluons are required
to make a classical field.

Transformation of a scalar field:

$$V(x) \;=\; \frac{q}{(x_T^2 + x_3^2)^{1/2}} \;=\; V'(x') \;=\; \frac{q}{(x_T^2 + \gamma^2 \Delta^2)^{1/2}}$$

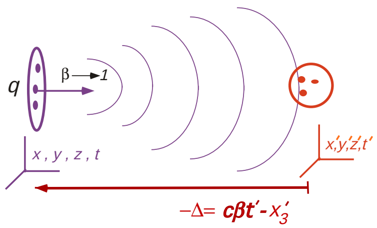
From the Lorentz transformation:

$$x_3 = \gamma(\beta c t' - x_3') \equiv -\gamma\Delta.$$

Closest approach is at $\Delta = 0$, i.e. $t' = \frac{1}{\beta c} x_3'$.

The scalar field transforms "like a ruler": At any fixed $\Delta \neq 0$, the field decreases like $1/\gamma = \sqrt{1-\beta^2}$.

Why? Because when the source sees a distance x_3, the observer sees a much larger distance.



field	x frame	x' frame
scalar	$\frac{q}{\lvert\vec{x}\rvert}$	$\frac{q}{(x_T^2+\gamma^2\Delta^2)^{1/2}} \sim \frac{1}{\gamma}$
gauge (0)	$A^0(x) = \frac{q}{\lvert\vec{x}\rvert}$	$A'^0(x') = \frac{-q\gamma}{(x_T^2+\gamma^2\Delta^2)^{1/2}} \sim \gamma^0$
field strength	$E_3(x) = \frac{q}{\lvert\vec{x}\rvert^2}$	$E'_3(x') = \frac{-q\gamma\Delta}{(x_T^2+\gamma^2\Delta^2)^{3/2}} \sim \frac{1}{\gamma^2}$

- **The "gluon field" A'^μ is enhanced, yet is a total derivative:**

$$A'^\mu = q \frac{\partial}{\partial x'_\mu} \ln\left(\Delta(t', x'_3)\right) + \mathcal{O}(1-\beta) \sim A'^-$$

- **The "large" part of A'^μ can be removed by a gauge transformation!**

- The electric, $\vec{\mathrm{E}}$ field of the incident particle does not overlap the "target" until the moment of the scattering.

- "Advanced" effects are corrections to the total derivative:

$$1-\beta \sim \frac{1}{2}\left[\sqrt{1-\beta^2}\right]^2 \sim \frac{m^2}{2E^2}$$

- Power-suppressed! These are corrections to factorization.

- At the same time, a gauge transformation also induces a phase on charged fields:

$$q(x) \Rightarrow q(x)\; e^{i\ln(\Delta)}$$

- Origin of the gauge links in factorized PDFs.

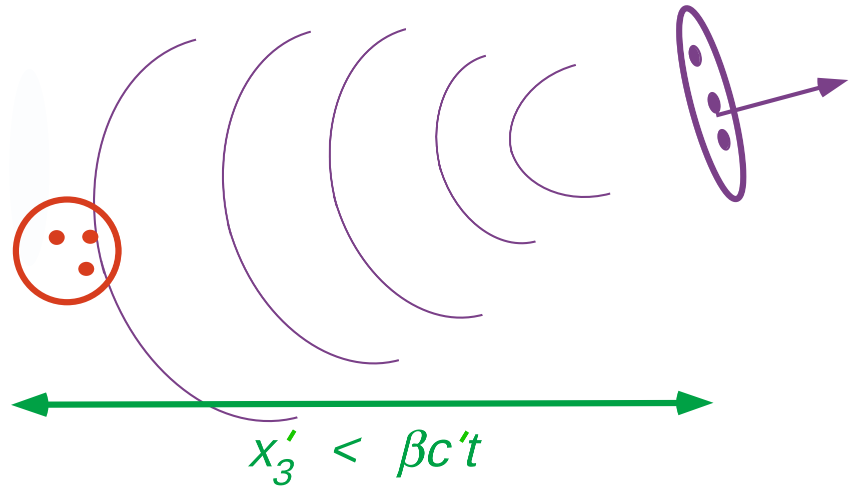
- **These phases cancel if the fields are well-localized $\Leftrightarrow$ σ inclusive.**

- **Initial-state interactions decouple from hard scattering**

- **Summarized by multiplicative factors: the parton distributions.**

$\Rightarrow$ Cross section for inclusive hard scattering is IR safe, with power-suppressed corrections.

- **What about final state interactions? Much of the same reasoning holds:**



- **Subtle but important difference: Δ changes sign in the final state.**
- **Then the gauge function in $\ln(\Delta)$ gets an imaginary part.**
- $q(x) \Rightarrow q(x)\, e^{i\,\ln(\Delta)}$ **not the same phase.**
- **Mismatch between initial- and final-state interactions: DIS/DY sign differences (Collins).**
- **Can be important for observables involving correlations in the final state. (Collins & Qiu)**

3. Collinear Factorization at Fixed Q_T

The transverse momentum distribution at order α_s.

Extend factorization to gluon radiation process:

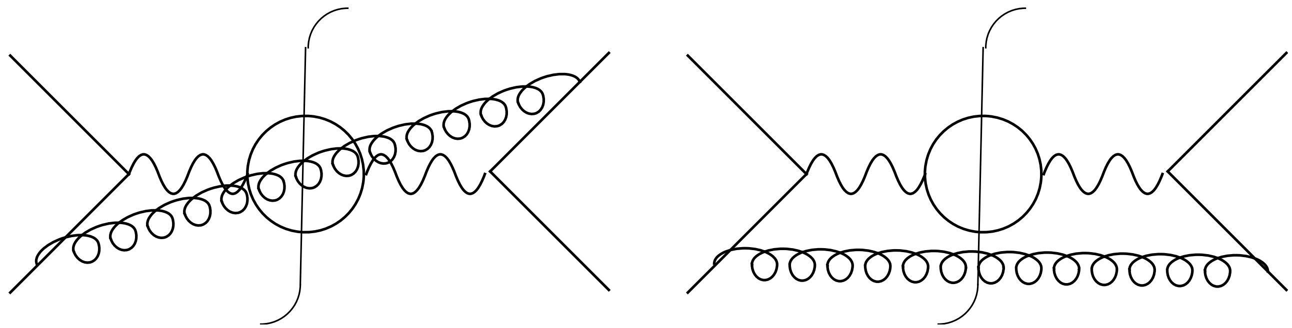
$$q(p_1) + \bar{q}(p_2) \to \gamma^*(Q) + g(k)$$

Treat this $2 \to 2$ process at lowest order (α_s) "LO" in factorized cross section, so that $\mathrm{k} = -\mathrm{Q}$.

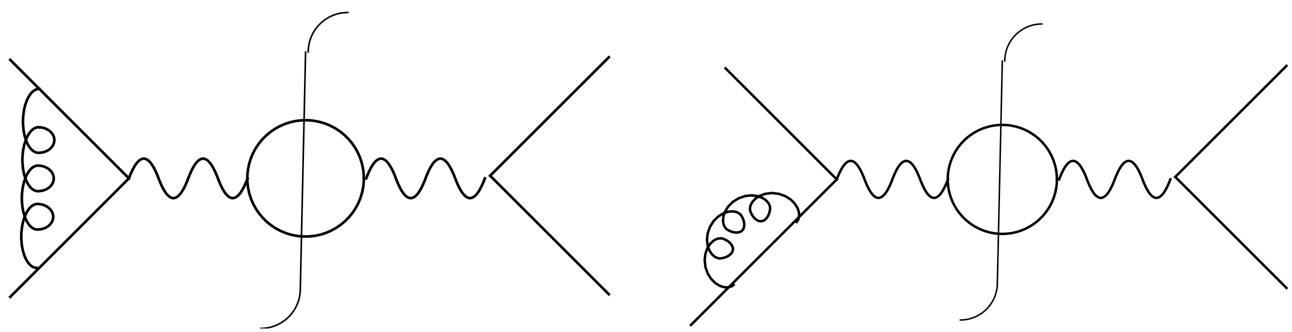
The result is well-defined for $\mathrm{Q}_T \neq 0$.

- **The diagrams at order α_s**

Gluon emission contributes at $Q_T \neq 0$



Virtual corrections contribute only at $Q_T = 0$



$$\frac{d^2\sigma^{(1)}_{q\bar{q}\to\gamma^* g}(z, Q^2, \mathrm{Q}_T)}{dQ^2\, d^2\mathrm{Q}_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2}\left(1 - \frac{4\mathrm{Q}_T^2}{(1-z)^2\hat{s}}\right)^{-1/2} \times \left[\frac{1}{\mathrm{Q}_T^2}\frac{1+z^2}{(1-\mathrm{z})} - \frac{2z}{(1-\mathrm{z})Q^2}\right]$$

Fine as long as $\mathrm{Q}_T \neq 0$, $z = Q^2/S \neq 1$.

Q_T integral $\to \ln(1-z)/(1-z)$, z integral $\to \ln(Q_T)/Q_T$.

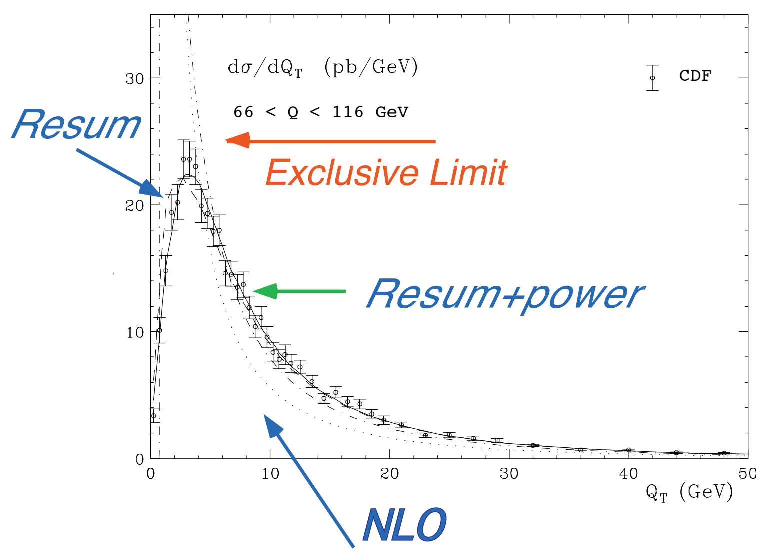
Both off these limits can be dealt with by reorganization, "resummation" of higher order corrections

4. TMD Factorization for Drell-Yan Production

- Q_T factorized cross sections: the motivation
- Low Q_T Drell-Yan & Higgs at leading log (LL) $({\alpha_s}^n \ln^{2n-1} Q_T)$

$$\frac{d\sigma(Q)}{dQ_T} \sim \frac{d}{dQ_T} \exp\left[-\frac{\alpha_s}{\pi} C_F \ln^2\left(\frac{Q}{Q_T}\right)\right]$$

$$(C_F = 4/3)$$



- **General features:**

  Maximum then decrease near "exclusive" limit (parton model kinematics) replaces divergence at $Q_T = 0$

  Soft but perturbative radiation broadens distribution

  Typically nonperturbative correction necessary for full quantitative description, esp. for $Q \sim$ few GeV.

  Recover fixed order predictions $\sigma^{(1)}$ away from $Q_T \ll Q$.

Getting to $Q_T \ll Q$: Transverse momentum resummation

- (Logs of Q_T)/Q_T to all orders

How? Variant factorization and separation of variables

q and $\bar{q}$ "arrive" at point of annihilation with transverse momentum of radiated gluons in initial state.

q and $\bar{q}$ radiate independently (fields don't overlap!).

Final-state QCD radiation too late to affect cross section

Summarized by: Q_T-factorization (Collins-Soper):

$$\frac{d\sigma_{NN\to QX}}{dQd^2Q_T}$$
$$= \int d\xi_1 d\xi_2 \; d^2\mathrm{k}_{1T} d^2\mathrm{k}_{2T} d^2\mathrm{k}_{sT} \, \delta\left(Q_T - k_{1T} - k_{2T} - k_{sT}\right)$$
$$\times \; H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})$$
$$\times U_{a\bar{a}}(k_{sT}, n)$$

The $\mathcal{P}'s$: new Transverse momentum-dependent TMD Distribution:

- **An operator definition**

$$\mathcal{P}_{f/H}(x, \mathrm{k}, p\cdot n, \epsilon) = \frac{1}{4N_C}\int \frac{d\lambda}{2\pi}\frac{d^2\mathrm{b}}{(2\pi)^2}\ \mathrm{e}^{-i\lambda x p\cdot u + i\mathrm{b}\cdot\mathrm{k}}$$

$$\times \langle H(p)|\bar{q}_f(0^+, \lambda, \mathrm{b})\gamma\cdot u\ q_f(0)|H(p)\rangle$$

- **In this case, the gauge links distinguish initial and final states as above: see new form by John Collins (As in Aybat and Rogers, 2011).**

Also need $U(b) = \int de \sigma^{(\mathrm{eik})}(\xi, Q, \mathrm{k}, \epsilon)$: "soft function" for wide-angle radiation

$$U_{a\bar{a}}(k_{sT}, n) = \int \frac{d^2\mathrm{b}}{(2\pi)^2}\, \mathrm{e}^{i\mathrm{b}\cdot\mathrm{k}_{sT}}$$
$$\times \frac{1}{d(c)}\, \mathrm{Tr}\, \langle 0|\, \bar{\mathrm{T}} \left[\mathcal{W}^{(c\bar{d})}(0)^{\dagger} \right] \mathrm{T} \left[\mathcal{W}^{(c\bar{d})}(\mathrm{b}) \right] |0\rangle\,.$$

$$\mathcal{W}^{(c\bar{d})}(X) = \Phi^{(\bar{d})}_{\beta'}(0, -\infty; X)\ \Phi^{(c)}_{\beta}(0, -\infty; X)\ ,$$

- Can be absorbed into the definition of the TMDs for one process: DY for example (Collins).

- **Symbolically:**

$$\frac{d\sigma_{NN\to QX}}{dQd^2Q_T} = H \times \mathcal{P}_{a/N}(\xi_1, p_1 \cdot n, k_{1T})\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2 \cdot n, k_{2T})$$
$$\otimes_{\xi_i, k_{iT}}\, U_{a\bar{a}}(k_{sT}, n)$$

We can *solve* **for the k_T dependence of the $\mathcal{P}$'s.**

New factorization variables: n^μ apportions gluons k:

$$p_i \cdot k < n \cdot k \;\Rightarrow\; k \in \mathcal{P}_i$$
$$p_a \cdot k, p_{\bar{a}} \cdot k > n \cdot k \;\Rightarrow\; k \in U$$

Convolution in $k_{i,T}$s $\Rightarrow$ Fourier $\mathrm{e}^{i\vec{Q}_T \cdot \vec{b}}$

• **The factorized cross section in "impact parameter space":**

$$\frac{d\sigma_{NN\to QX}(Q,b)}{dQ}$$

$$= \int d\xi_1 d\xi_2\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$

$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, b)\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, b)\ U_{a\bar{a}}(b,n)$$

Now we can solve for b dependence just by separating variables!

the LHS independent of $\mu_{\rm ren}$, n $\Rightarrow$ two equations

$$\mu_{\rm ren}\frac{d\sigma}{d\mu_{\rm ren}} = 0 \quad n^\alpha\frac{d\sigma}{dn^\alpha} = 0$$

- **Solve and transform back to Q_T: all the (Logs of Q_T)/Q_T:**

$$\frac{d\sigma_{NN\mathrm{res}}}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}} \exp\left[\, E_{a\bar{a}}^{\mathrm{PT}}(b,Q,\mu)\,\right]$$

$$\times \sum_{a=q\bar{q}} \int \xi_1 \xi_2\, \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\; f_{a/N}(\xi_1, 1/b)\; f_{\bar{a}/N}(\xi_2, 1/b)$$

"Sudakov" exponent suppresses large $b \leftrightarrow$ small Q_T:

$$E_{a\bar{a}}^{\mathrm{PT}} = -\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right) + 2B_q(\alpha_s(k_T)) \right]$$

With $B = 2(K+G)_{\mu=p\cdot n}$, and lower limit: $1/b$ (NLL)

- **Comments:**

  The functions $A_i(\alpha_s)$ and $B_i(\alpha_s)$ are anomalous dimensions.

  And can be calculated by comparison to low orders.

  In particular, $A_i(\alpha_s)$ is the numerator of the $1/(1-x)$ term in splitting function $P_{ii}(x)$

  because it's the infrared divergent $(x \to 1)$ coefficient of the collinear $b \to \infty$ singularity.

- $A_q(\alpha_s) = \frac{\alpha_s}{\pi} C_q \left(1 + \frac{\alpha_s}{\pi} K + \ldots\right)$,

$$K = C_A \left(\frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{5 n_F}{9}$$

- **Evaluating a resummed cross sections: re-enter NPQCD.**

We start with:

$$E^{\rm PT} = - \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T)) \ln\left(\frac{Q^2}{k_T^2}\right) + B_q(\alpha_s(k_T)) \right]$$

With running coupling:

$$\alpha_s(k_T) = \frac{\alpha_s(Q)}{1 + \frac{\alpha_s(Q)}{4\pi}\beta_0 \ln\left(\frac{k_T^2}{Q^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{k_T^2}{\Lambda_{\rm QCD}^2}\right)}$$

Singularity in integral at

$$b^2 = Q^2 \exp[-4\pi/\beta_0\alpha_s(Q)] \sim \frac{1}{\Lambda^2}.$$

- **Problem:** how to do the inverse transform with the running coupling when $k_T^{\min} \sim 1/b$ gets small?

- A whole bunch of approaches:

1) Work in Q_T-space directly to some approximation
The originals: Dokshitzer, Diakanov & Troyan
Revived by Ellis & Veseli Kulesza & Stirling
who re-derived it from b-space.

2) Insert a "soft landing" on the k_T integral by replacing

$$1/b \to \sqrt{1/b^2 + 1/b_*^2}$$

for some fixed b_*. (CS, CSS "b_*" prescription, ResBos)

3) Extrapolation of $E^{\rm PT}$ into NP region (Qiu, Zhang).

4) Minimal: avoid the singularity at $1/b = \Lambda_{\rm QCD}$
by monkeying with the b-space contour integral.
(This technique introduced in threshold resummation;
then adapted by Laenen, GS and Vogelsang,
and Bozzi, Catani, de Florian and Grazzini.)

- 5) Effective theory (SCET) treatments (Stewart, Tackmann, Waalewijn et al, Becher and Neubert, Petriello and Mantry). Multi-step evolution in momentum rather than moment/impact parameter spaces.

Any of these “define” PT. All will fit the data qualitatively, and with a little work quantitatively.

But at low Q_T require new parameters for quantitative fit.
This is not all bad . . . let’s see why.

- **Window to nonperturbative distributions:**

$$\begin{aligned} E^{\rm soft} &= \frac{1}{2\pi}\int_0^{\mu_I^2}\frac{d^2k_T}{k_T^2}\,A_q(\alpha_s(k_T))\,\ln\left(\frac{Q^2}{k_T^2}\right)\left(e^{i\mathrm{b}\cdot\mathrm{k}_T}-1\right) \\ &\sim -\int_0^{\mu_I^2}\frac{dk_T^2}{k_T^2}\,(\mathrm{b}\cdot\mathrm{k}_T)^2 A_q(\alpha_s(k_T))\,\ln\left(\frac{Q^2}{k_T^2}\right)+\cdots \\ &\sim -\,b^2\int dk_T^2\,A_q(\alpha_s(k_T))\,\ln\left(\frac{Q^2}{k_T^2}\right) \end{aligned}$$

$\theta(k_T - 1/b) \Rightarrow (e^{i\mathrm{b}\cdot\mathrm{k}_T} - 1)$; **in fact, correct to all orders,**

Note the expansion is for b " small enough" only.

What is $-\, b^2 \int dk_T^2\, A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right)$ **?**

- **Related to** $dU(b)/db^2$ **and**
- **Suggests a nonperturbative correction of the form (exhibiting the** μ_I **is unconventional)**

$$E^{\mathrm{NP}} = -\, b^2 \mu_I^2 \left(g_1 \ln\left(\frac{Q}{\mu_I}\right) + g_2 \right)$$

Since this is an exponent, whatever the definition of the pertrubative resummed cross section, it is smeared with a Gaussian whose width in b **(**k_T**) space decreases (increases) with** $\ln Q$**.**

In summary

$$\frac{d\sigma(Q_T)}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2 b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}}\, e^{E^{\rm PT}_{a\bar{a}}(b,Q,\mu)}$$

$$\times \quad e^{-\ \mu_I^2 b^2 (g_1\ \ln\left(\frac{Q}{\mu_I}\right)+g_2)}$$

$$\times \sum_{a=q\bar{q}} \int \xi_1 \xi_2\, \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}$$

$$\times \quad f_{a/N}(\xi_1, 1/b)\, f_{\bar{a}/N}(\xi_2, 1/b)$$

$$= \pi \int d^2 \mathrm{k}_T\, \frac{e^{-k_T^2/4[\mu_I^2(g_2 \ln(Q/k_T)+g_2)]}}{\mu_I^2(g_2 \ln(Q/k_T) + g_2)}\, \frac{d\sigma_{NN}(\mathrm{Q_T} - \mathrm{k_T})}{dQ^2\, d^2\vec{Q}_T}$$

Which gives curves like the one we saw before.

6. A Few concluding Comments

- Factorization in quantum field theory is closely related to classical considerations.
- Differences between initial- and final-state gauge links are consistent with this factorization.
- There is a well-developed theory of factorization for Drell-Yan, including transverse momentum dependence.
- The 'QCD-inclusive' nature of Drell-Yan production maintains the underlying factorization.
- Nonperturbative effects play an essential role at low Q_T and should be thought of as an integral part of the formalism.
- The stage is set for a new phenomenology to explore the transverse-momentum dependent and spin-sensitive parton distributions.