Propagation of Partially Correlated Statistical Errors

Lara De Nardo

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Let's suppose we want to compare some quantity, like an asymmetry, calculated with and without some correction, like an alignment correction. Since the alignment correction changes the value of θ and φ, events will migrate from each kinematic bin to the neighboring ones, and, also, from the neighboring bins into the given bin.

We can denote with $A_{||}$ and $A'_{||}$ the asymmetries before and after the correction, with:

$$A_{||} = \frac{A-B}{A+B} \qquad\qquad A'_{||} = \frac{A'-B'}{A'+B'} \; , \tag{1}$$

where A and B denote for example the yields for two target states ($A = N_A/L_A$, and $B = N_B/L_B$, with N and L the number of events and the luminosities in the given target state).

If, in order to compare them, we are interested in calculating the difference or the ratio of the two asymmetries, then in the formula for the error propagation we have to include a term[1] which looks like

$$cov(A_{||}, A'_{||}) \; . \tag{2}$$

This covariance term can be written as:

$$\begin{aligned} & cov\left(\frac{A-B}{A+B}, \frac{A'-B'}{A'+B'}\right) = \\ & cov\left(\frac{A}{A+B}, \frac{A'}{A'+B'}\right) - cov\left(\frac{B}{A+B}, \frac{A'}{A'+B'}\right) - \\ & -cov\left(\frac{A}{A+B}, \frac{B'}{A'+B'}\right) + cov\left(\frac{B}{A+B}, \frac{B'}{A'+B'}\right) \; . \end{aligned} \tag{3}$$

Each of these terms is easily calculable if one considers that:

$$cov(f(A,B),C) = \frac{\partial f}{\partial A} cov(A,C) + \frac{\partial f}{\partial B} cov(B,C) \; . \tag{4}$$

One then obtains:

$$\begin{aligned} & cov\left(\frac{A}{A+B}, \frac{A'}{A'+B'}\right) = \\ & \frac{B}{(A+B)^2}\frac{B'}{(A'+B')^2} cov(A,A') - \frac{B}{(A+B)^2}\frac{A'}{(A'+B')^2} cov(A,B') - \\ & - \frac{A}{(A+B)^2}\frac{B'}{(A'+B')^2} cov(B,A') + \frac{A}{(A+B)^2}\frac{A'}{(A'+B')^2} cov(B,B') \; , \end{aligned} \tag{5}$$

[1] See internal note 02-008.

$$cov\left(\frac{B}{A+B},\frac{A'}{A'+B'}\right) =$$
$$\frac{A}{(A+B)^2}\frac{B'}{(A'+B')^2}cov(B,A') - \frac{A}{(A+B)^2}\frac{A'}{(A'+B')^2}cov(B,B') -$$
$$-\frac{B}{(A+B)^2}\frac{B'}{(A'+B')^2}cov(A,A') + \frac{B}{(A+B)^2}\frac{A'}{(A'+B')^2}cov(A,B') \text{ ,} \quad (6)$$

$$cov\left(\frac{A}{A+B},\frac{B'}{A'+B'}\right) =$$
$$\frac{B}{(A+B)^2}\frac{A'}{(A'+B')^2}cov(A,B') - \frac{A}{(A+B)^2}\frac{A'}{(A'+B')^2}cov(B,B') -$$
$$-\frac{B}{(A+B)^2}\frac{B'}{(A'+B')^2}cov(A,A') + \frac{A}{(A+B)^2}\frac{B'}{(A'+B')^2}cov(B,A') \text{ ,} \quad (7)$$

$$cov\left(\frac{B}{A+B},\frac{B'}{A'+B'}\right) =$$
$$\frac{A}{(A+B)^2}\frac{A'}{(A'+B')^2}cov(B,B') - \frac{A}{(A+B)^2}\frac{B'}{(A'+B')^2}cov(B,A') -$$
$$-\frac{B}{(A+B)^2}\frac{A'}{(A'+B')^2}cov(A,B') + \frac{B}{(A+B)^2}\frac{B'}{(A'+B')^2}cov(A,A') \text{ .} \quad (8)$$

The result of eq.(3) is

$$cov\left(\frac{A-B}{A+B},\frac{A'-B'}{A'+B'}\right) =$$
$$\frac{4}{(A+B)^2(A'+B')^2}\left[BB'\,cov(A,A') + AA'\,cov(B,B') -\right.$$
$$\left. -\,AB'\,cov(B,A') - BA'\,cov(A,B')\right] \quad (9)$$

In the most common cases the quantities A and B are not correlated, as in the case in which they are calculated for two different target states, and as a consequence the last two terms are zero.

The error of $A_{||} - A'_{||}$ becomes:

$$\sigma^2 = \sigma^2_{A_{||}} + \sigma^2_{A'_{||}} - 2\,cov(A_{||},A'_{||}) =$$
$$\sigma^2_{A_{||}} + \sigma^2_{A'_{||}} - \frac{8}{(A+B)^2(A'+B')^2}\left[BB'\,cov(A,A') + AA'\,cov(B,B')\right] \quad (10)$$

The problem is then reduced to calculating the correlation between A (B) before and after the correction.

For partially correlated variables we know that:

$$cov(A,A') = \frac{\sigma^2_A \sigma^2_{A'}}{\sigma^2_{A\cap A'}} \text{ .} \quad (11)$$

In the case in which A can be expressed as $A = N_A/L_A$, one has that

$$\sigma_A^2 = \frac{N_A}{L_A^2} \quad , \quad \sigma_{A'}^2 = \frac{N'_A}{L_A^2} \quad , \quad \sigma_{A\cap A'}^2 = \frac{N_A - N_A^{out}}{L_A^2} \tag{12}$$

where N_A^{out} are the events that have migrated *out* of the given bin[2]. One then obtains:

$$cov(A, A') = \frac{1}{L_A^2} \frac{N_A N_{A'}}{N_A - N_A^{out}} \ . \tag{13}$$

[2]Note that $N'_A = N_A - N_A^{out} + N_A^{in}$