Kinematics of electron–proton collisions

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Kinematics of electron–proton collisions are presented for massless and massive parton constituents and outgoing leptons.

On présente la cinématique des collisions électrons–protons pour des partons constituants et des leptons émis sans masse et avec masse.

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I. Introduction

In this paper we consider the kinematics of the process e + P → L + X. In the parton model (1), the hadronic final state X consists of two separate jets of collinear hadrons: these jets of hadrons are expected to come from the struck parton (J) and from the fragments of the proton (T), respectively (Fig. 1). In order to maintain as close a correspondence as possible to the formulae (and intuition) of fixed target leptoproduction, we will use the following conventions: (*1*) all angles are taken with respect to the direction of the incident electron beam, and (*2*) longitudinal components of momenta are positive in the incident electron direction. The struck-parton jet J will in general be at an azimuthal angle of 180° with respect to the outgoing lepton L. We will adopt the convention that the outgoing lepton L and the jet J *both* have positive transverse momenta; we must implicitly remember that this positivity refers to J and L coming out at a 180° azimuthal separation.

In Sect. II we derive expressions for the outgoing lepton 4-momentum in terms of useful invariants of the scattering process. Results are obtained for massive as well as massless outgoing leptons. In Sect. III we express hadron jet momenta in terms of these same invariants by assuming the electron scatters off a massless parton. These results are generalized in Sect. IV to include the possibility of massive incoming and outgoing partons, as well as a massive outgoing lepton. Surprisingly, we find that the kinematic scaling variable ξ for massive partons is *not* the same as that obtained by more sophisticated techniques (2, 3). The origin of this misconception is discussed in detail.

The kinematics of the "virtual-photon", proton center-of-mass (γ^*P c.m.) frame are considered in Sect. V. The Lorentz transformation between lab and γ^*P c.m. frames is explicitly constructed, as quantum chromodynamics (QCD) predictions for angular energy flow (4), hadronic azimuthal asymmetries (5), Sterman–Weinberg jet angle parametrizations (6), and for the "event shape" structure of hadronic final states (7) in electron–proton scattering have all been worked out in the γ^*P c.m. frame.

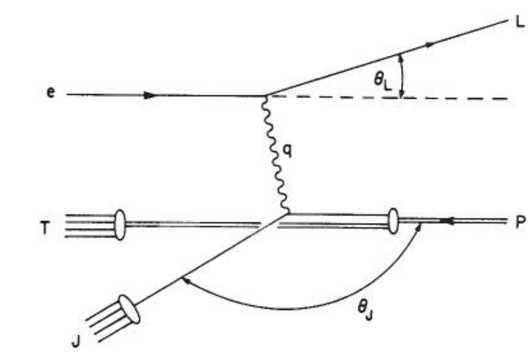

FIG. 1. Laboratory frame scattering angles for electron–proton collisions.

The magnitude of mass effects at proposed CHEER beam energies is briefly considered in our concluding section.

II. Lepton Observables and Useful Invariants

Consider the reaction e + P → L + X, where L is a charged or neutral lepton of mass m_L and X is the hadronic final state. The electron, proton, and the final-state lepton L have 4-momenta given by

$$[1] \quad \text{e:} \quad \mathfrak{p}_e = (E_e, 0, 0, E_e)$$

$$[2] \quad \text{P:} \quad \mathfrak{P} = (E_P, 0, 0, -P_P)$$

$$[3] \quad \text{L:} \quad \mathfrak{p}_L = (E_L, p_L \sin\theta_L, 0, p_L \cos\theta_L)$$

The parameters E_L, p_L, and θ_L are determined from

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measurement of the final-state lepton trajectory; E_e, E_P, P_P, E_L, p_L, and θ_L are a set of known parameters (E_e, E_P, E_L, and θ_L in the limit m_P and m_L are small).

We seek to construct reaction invariants from these parameters in order to provide a framework for hadron jet kinematics. The square of the invariant mass of the interaction $e + P \to L + X$ is denoted by s, where

$$[4] \quad s \equiv (\mathfrak{p}_e + \mathfrak{P})^2 = 2E_e(E_P + P_P) + m_P{}^2$$

For high energies, E_P, $P_P \gg m_P$, and $\sqrt{s}$ is just $4E_eE_P$. The invariant Q^2 is defined to be minus the square of momentum transfer from leptons to hadrons:

$$[5] \quad Q^2 \equiv -(\mathfrak{p}_e - \mathfrak{p}_L)^2 = -E_L{}^2 + p_L{}^2 + 2E_eE_L - 2E_ep_L\cos\theta_L$$

The energy transferred to the hadronic final state in the proton rest frame is defined to be ν, where

$$[6] \quad m_P\nu \equiv \mathfrak{P}\cdot(\mathfrak{p}_e - \mathfrak{p}_L) = E_P(E_e - E_L) + P_P(E_e - p_L\cos\theta_L)$$

Finally, the square of the total invariant mass available to hadrons is given by

$$[7] \quad W^2 = (\mathfrak{P} + \mathfrak{p}_e - \mathfrak{p}_L)^2 = m_P{}^2 - \{-E_L{}^2 + p_L{}^2 + 2E_eE_L - 2E_ep_L\cos\theta_L\} + 2\{E_P(E_e - E_L) + P_P(E_e - p_L\cos\theta_L\} = 2m_P\nu - Q^2 + m_P{}^2$$

In the limit $E_L = p_L$, $E_P = P_P$, corresponding to lepton and proton masses which are negligible compared to their energies, [4]–[6] become

$$[8] \quad s \simeq 4E_eE_P + m_P{}^2(1 - E_e/E_P)$$

$$[9] \quad Q^2 \simeq 4E_eE_L\sin^2(\theta_L/2)$$

$$[10] \quad m_P\nu \simeq 2E_P(E_e - E_L\cos^2(\theta_L/2))$$

Note that even if m_L is not light ($E_L \neq p_L$), [7] and the relation

$$[11] \quad m_P\nu_{\max} = 2E_eE_P = s - m_P{}^2$$

will still be true. All lepton mass dependence of W^2 and s is implicit in Q^2 and ν.

Let us now define scaling variables x and y by

$$[12] \quad x \equiv Q^2/2m_P\nu$$

$$[13] \quad y \equiv \nu/\nu_{\max} = m_P\nu/2E_eE_P$$

where Q^2 and ν are given in terms of lepton observables by [5] and [6]. It will prove useful for us to express lepton observables in terms of x and y. Clearly,

$$[14] \quad Q^2 = 4E_eE_Pxy$$

and

$$[15] \quad m_P\nu = 2E_eE_Py$$

Using [9] and [10], we find that

$$[16] \quad \sin^2(\theta_L/2) = E_Pxy/E_L$$

$$[17] \quad \cos^2(\theta_L/2) = E_e(1 - y)/E_L$$

in which case we see that

$$[18] \quad E_L = E_e(1 - y) + E_Pxy$$

$$[19] \quad \tan^2(\theta_L/2) = E_Pxy/[E_e(1 - y)]$$

Equation [18] can be substituted into [16] and [17] in order to find that

$$[20] \quad \cos\theta_L = 2\cos^2(\theta_L/2) - 1 = [E_e(1 - y) - E_Pxy] \div [E_e(1 - y) + E_Pxy]$$

$$[21] \quad \sin\theta_L = 2[xy(1 - y)E_eE_P]^{1/2} \div [E_e(1 - y) + E_Pxy]$$

Finally we see that the denominators of [20] and [21] are the lepton energy, [18]. Consequently, the respective numerators give parallel and perpendicular components of lepton momentum:

$$[22] \quad p_{L_\parallel} = E_e(1 - y) - E_Pxy$$

$$[23] \quad p_{L_\perp} = [4xy(1 - y)E_eE_P]^{1/2}$$

Equations [18], [22], and [23] are appropriate for the limit that all momenta are large compared to the proton and outgoing lepton masses m_P and m_L. Suppose we no longer disregard these masses – a massive lepton could be "electroproduced" either through an anomalous nondiagonal neutral current interaction or else through charged current coupling to a heavy neutral lepton. We find from [5] and [6] that

$$[24] \quad E_L = E_e + [1/(E_P + P_P)] \times [P_P(Q^2 + m_L{}^2)/(2E_e) - m_P\nu]$$

$$[25] \quad p_{L_\parallel} = p_L\cos\theta_L = E_e - [1/(E_P + P_P)] \times [E_P(Q^2 + m_L{}^2)/(2E_e) + m_P\nu]$$

in which case

$$[26] \quad (p_{L_\perp})^2 = (p_L\sin\theta_L)^2 = E_L{}^2 - (p_{L_\parallel})^2 - m_L{}^2 = Q^2 - \{(Q^2 + m_L{}^2)/[E_e(E_P + P_P)]\} \times \{m_P\nu + m_P{}^2(Q^2 + m_L{}^2)/[4E_e(E_P + P_P)]\}$$

Of course, [24]–[26] are equivalent to [18], [22], and [23] in the limit $m_L = m_P = 0$.

III. Hadron Jet Kinematics for Massless Partons

The kinematical relations which we have derived up to this point have involved only the lepton observables. We have defined scaling variables in terms of these observables and then derived model independent relations between the observables.

In order to discuss the final state of the struck hadron, on the other hand, we must now appeal to some model of hadronic structure. This is because the hadron, unlike the lepton, is a composite, non-pointlike object which will be torn apart in the collisions of interest to us.

For simplicity (and with an eye on its astonishing degree of success) we shall adopt the parton model of Feynman (1). In this model the nucleon is considered to be (at sufficiently large momentum) a collection of (approximately) free, on-mass-shell constituents which share its momentum. Thus, deep inelastic scattering is the elastic scattering of an electron with one of the constituents. This is just the impulse approximation (1). As we show below, the fact that the initial (incident) and final (scattered) parton are on-mass-shell will serve to fix the hadron kinematics. We will assume that the outgoing parton fragments into a (more or less) collimated jet of hadrons sharing its momentum (Fig. 1). Consequently, we expect the hadron final state X in $e + P \rightarrow L + X$ to consist of two jets J and T, J corresponding to the struck parton, and T corresponding to target fragments (Fig. 1).

We begin by considering the scattering of an electron from a parton carrying a fraction ξ of the proton momentum. The 4-momentum of the parton is given by

$$[27] \quad \mathfrak{p}_i \equiv \xi\mathfrak{P} = (\xi E_P, 0, 0, -\xi E_P)$$

where both parton and nucleon masses are neglected compared to E_P. The 4-momentum of the outgoing parton is given by $\mathfrak{p}_f \equiv (\xi\mathfrak{P} + \mathfrak{p}_e - \mathfrak{p}_L)$, where $\mathfrak{p}_e$ and $\mathfrak{p}_L$ are given by [1] and [3], and where we ignore the outgoing lepton mass ($p_L = E_L$). The requirement that the outgoing parton be on-mass-shell allows us to find ξ in terms of x and y. First note that

$$[28] \quad 0 = \mathfrak{p}_f^{\,2} = (\xi\mathfrak{P} + \mathfrak{p}_e - \mathfrak{p}_L)^2 = 4\xi E_e E_P - 2E_e E_L(1 - \cos\theta_L) - 2\xi E_L E_P(1 + \cos\theta_L)$$

Substituting [18] and [20] into [28], we find that the right-hand side becomes $4E_e E_P(-xy + \xi y)$, which vanishes provided

$$[29] \quad \xi = x$$

We conclude that the struck parton must carry a fraction of the proton momentum given by $x = Q^2/2m_P\nu$ in order for the parton subprocess of Fig. 1 to remain on-shell. Of course, x can be obtained from the lepton observables E_L and θ_L (see [9], [10], and [12]).

Using this result, we may now determine the momentum of the outgoing "current" jet (J) of hadrons. The transverse component of the jet momentum $p_{J_\perp}$ must balance that of the outgoing lepton. We take the convention that positive hadron $p_\perp$ is opposite to positive lepton $p_\perp$ (see [23]); therefore,

$$[30] \quad p_{J_\perp} = p_L \sin\theta_L = [4xy(1 - y)E_e E_P]^{1/2}$$

$$[31] \quad p_{J_\parallel} = E_e - xE_P - E_L\cos\theta_L = yE_e - (1 - y)xE_P$$

If hadrons in the current jet all have small masses compared to the jet momentum, then the energy of the current jet is given by

$$[32] \quad E_J \simeq [p_{J_\perp}{}^2 + p_{J_\parallel}{}^2]^{1/2} = yE_e + (1 - y)xE_P$$

The angle θ_J that the current jet makes with respect to the incoming electron (Fig. 1) is given by

$$[33] \quad \cos\theta_J = p_{J_\parallel}/E_J = [yE_e - (1 - y)xE_P] \div [yE_e + (1 - y)xE_P]$$

$$[34] \quad \tan^2(\theta_J/2) = (1 - \cos\theta_J)/(1 + \cos\theta_J) = (1 - y)xE_P/(yE_e)$$

Note that expressions for the outgoing lepton and current jet momenta are related by an interchange of y and $(1 - y)$, and that the expressions satisfy energy–momentum conservation:

$$[35] \quad E_L + E_J = E_e + xE_P$$

$$[36] \quad p_{L_\parallel} + p_{J_\parallel} = E_e - xE_P$$

$$[37] \quad p_{L_\perp} + p_{J_\perp} = 0$$

Finally we note that the proton target fragments T (Fig. 1) remain unscattered:

$$[38] \quad \mathfrak{P}_T = (1 - x)\mathfrak{P} = [(1 - x)E_P, 0, 0, -(1 - x)E_P]$$

Consequently, the *total* longitudinal momentum, transverse momentum, and energy of the hadronic final state X is given by summing jet, [30]–[32], and target, [38], 4-momenta:

$$[39] \quad \sum_{\text{hadrons}} E_h = E_e y + E_P(1 - xy)$$

$$[40] \quad \sum_{\text{hadrons}} (p_\parallel)_h = E_e y - E_P(1 - xy)$$

$$[41] \quad \sum_{\text{hadrons}} [(p_\perp)_h]^2 = 4E_e E_P xy(1 - y)$$

IV. Kinematics for Massive Scattering Constituents

So far we have assumed that the proton, the incident parton, and the outgoing parton are all massless. Although this approximation is appropriate for most values of ν and Q^2 for present beam energies, there remain parameter domains which are sensitive to these masses. We choose to parameterize this sensitivity by redefining ξ to be the ratio of parton-to-proton energy-plus-momentum (2):

$$[42] \quad \xi \equiv (E_i + p_i)/(E_P + P_P)$$

The proton 4-momentum $\mathfrak{P}$ is given by [2], and the parton 4-momentum is given by

$$[43] \quad \mathfrak{p}_i = (E_i, 0, 0, -p_i)$$

such that $\mathfrak{P}^2 = m_P{}^2$ and $\mathfrak{p}_i{}^2 = m_i{}^2$, the squared masses of the proton and incoming parton, respectively. If the outgoing parton is on-mass-shell ($\mathfrak{p}_f{}^2 = m_f{}^2$), then

$$[44] \quad m_f{}^2 = (\mathfrak{p}_i + \mathfrak{p}_e - \mathfrak{p}_L)^2 = m_i{}^2 + 2\mathfrak{p}_i \cdot \mathfrak{q} - Q^2$$

where $\mathfrak{q} \equiv \mathfrak{p}_e - \mathfrak{p}_L$ (see [1] and [3]). A little algebraic manipulation transforms [44] into a quadratic equation in the variable $[(E_i + p_i)(q_0 + q_3)]$:

$$[45] \quad 2\mathfrak{p}_i \cdot \mathfrak{q} = 2(E_i q_0 + p_i q_3) = [(E_i + p_i)(q_0 + q_3)] + (E_i - p_i)(q_0 - q_3)$$
$$= [(E_i + p_i)(q_0 + q_3)] - \{m_i{}^2(Q^2 - q_1{}^2)/[(E_i + p_i)(q_0 + q_3)]\}$$

in which case

$$[46] \quad [(E_i + p_i)(q_0 + q_3)] = \tfrac{1}{2}\{Q^2 + m_f{}^2 - m_i{}^2 + [(Q^2 + m_f{}^2 - m_i{}^2)^2 + 4m_i{}^2(Q^2 - q_1{}^2)]^{1/2}\}$$

We have taken the positive root, as $q_0 + q_3$ is positive definite.[1] Note that q_1 is just equal to $p_{L_\perp}$ (see [26]), the transverse component of the outgoing lepton momentum.

To obtain an expression for ξ, [42], we take the ratio of $[(E_i + p_i)(q_0 + q_3)]$ to $[(E_P + P_P)(q_0 + q_3)]$. This latter quantity can be expressed entirely in terms of m_P, Q^2, and ν as follows:

$$[47] \quad m_P\nu = \mathfrak{P} \cdot \mathfrak{q} = \tfrac{1}{2}\{[(E_P + P_P)(q_0 + q_3)] + (E_P - P_P)(q_0 - q_3)\}$$
$$= \tfrac{1}{2}\{[(E_P + P_P)(q_0 + q_3)] - m_P{}^2(Q^2 - q_1{}^2)/[(E_P + P_P)(q_0 + q_3)]\}$$

in which case

$$[48] \quad [(E_P + P_P)(q_0 + q_3)] = m_P\nu\{1 + [1 + (Q^2 - q_1{}^2)/\nu^2]^{1/2}\}$$

Substituting [46] and [48] into [42], we find that

$$[49] \quad \xi = \frac{Q^2 + m_f{}^2 - m_i{}^2 + [(Q^2 + m_f{}^2 - m_i{}^2)^2 + 4m_i{}^2(Q^2 - q_1{}^2)]^{1/2}}{2m_P\nu\{1 + [1 + (Q^2 - q_1{}^2)/\nu^2]^{1/2}\}}$$

where $(Q^2 - q_1{}^2) = Q^2 - p_{L_\perp}{}^2 = (Q^2 + m_L{}^2)\{m_P\nu + m_P{}^2(Q^2 + m_L{}^2)/[4E_e(E_P + P_P)]\}/[E_e(E_P + P_P)]$. Note that our derivation of [49] applies for a massive or massless outgoing lepton (provided Q^2 and $m_P\nu$ are given by [5] and [6]) and that $\xi \to x$ in the limit all masses vanish.

Equation [49] differs from the "standard" expression for ξ,

$$[50] \quad \xi = \frac{Q^2 + m_f{}^2 - m_i{}^2 + [(Q^2 + m_f{}^2 - m_i{}^2)^2 + 4m_i{}^2Q^2]^{1/2}}{2m_P\nu[1 + (1 + Q^2/\nu^2)^{1/2}]}$$

obtained from either the short distance operator product expansion or from the light cone analysis of current commutators (3). Although our definition, [42], and kinematic derivation of ξ correspond to Frampton's parton-model calculation (2), Frampton obtained [50] because he assumed that $(q_0 + q_3)(q_0 - q_3) = -Q^2$, thereby neglecting the perpendicular component of the outgoing lepton's momentum ($q_1 = -p_{L_\perp}$). Consequently, the statement (frequent in the literature) that parton model kinematics *alone* lead to *exactly* the same scaling variable as more sophisticated treatments is *incorrect*. Note, however, that [49] does *not* contradict the standard expression $\xi = Q^2/\{2m_P\nu[1 + (1 + Q^2/\nu^2)^{1/2}]\}$ obtained by assuming that $m_f = 0$ and that $\mathfrak{p}_i = \xi\mathfrak{P}$. The expressions can be reconciled once one realizes that for the latter expression $m_i{}^2 = \xi^2\mathfrak{P}^2 = (\xi m_P)^2$.

We now wish to obtain an expression for jet momenta and the jet angle θ_J for nonzero values of the four masses m_i, m_f, m_P, and m_L. First note from [42] that $E_i + p_i = \xi(E_P + P_P)$, $E_i - p_i = m_i{}^2/[\xi(E_P + P_P)]$,

[1] See [53] and [54] below.

in which case

[51] $E_i = \frac{1}{2}\{\xi(E_P + P_P) + m_i^2/[\xi(E_P + P_P)]\}$

[52] $p_i = \frac{1}{2}\{\xi(E_P + P_P) - m_i^2/[\xi(E_P + P_P)]\}$

where ξ is given *by [49]*. We obtain the components of the momentum transfer 4-vector $\mathfrak{q}$ from [24]–[26]:

[53] $q_0 = E_e - E_L = [(-P_P/2E_e)(Q^2 + m_L^2) + m_P\nu]/(E_P + P_P)$

[54] $q_3 = E_e - p_L \cos\theta_L = [(E_P/2E_e)(Q^2 + m_L^2) + m_P\nu]/(E_P + P_P)$

The transverse component q_1 is just $-p_{L_\perp}$ (equation [26]).

The outgoing parton 4-momentum $\mathfrak{p}_f$ has components corresponding to the energy and momentum of the current jet J:

[55] $p_{J_\parallel} = -p_i + q_3 = -\xi(E_P + P_P)/2 + [m_P\nu + E_P(Q^2 + m_L^2)/2E_e + m_i^2/2\xi]/(E_P + P_P)$

[56] $E_J = E_i + q_0 = \xi(E_P + P_P)/2 + [m_P\nu - P_P(Q^2 + m_L^2)/2E_e + m_i^2/2\xi]/(E_P + P_P)$

These expressions reduce to [31] and [32] when m_i, m_L, and m_P (but not $m_P\nu \equiv \mathfrak{P}\cdot\mathfrak{q}$) go to zero, as $\xi \to x$ in this limit (see [49]). The perpendicular component of jet momentum $p_{J_\perp}$ is just the positive square root of the right-hand side of [26], and

[57] $\tan^2\theta_J = (p_{L_\perp})^2/(p_{J_\parallel})^2$

V. The γ*P Center-of-mass Frame

A great deal of QCD phenomenology relevant to deep inelastic scattering has been worked out in the frame moving with the center-of-mass of the proton and the "virtual photon" (leptonic momentum transfer), hereafter denoted as the "γ*P c.m. frame" (4–7). In this section we develop the Lorentz transformation taking any lab frame 4-momenta into the γ*P c.m. frame. This transformation could be applied, for example, to the momenta of hadrons produced at CHEER in order to determine empirical "event shape" moments which could then be compared to published predictions (7).

We shall assume that parton, proton, and lepton masses can be neglected; proton, electron, and outgoing lepton 4-momenta in the lab frame are then given by

[58] $\mathfrak{P} = (E_P, 0, 0, -E_P)$

[59] $\mathfrak{p}_e = (E_e, 0, 0, E_e)$

[60] $\mathfrak{p}_L = (E_L, E_L \sin\theta_L, 0, E_L \cos\theta_L)$

The dimensionless variables x and y are obtained for measured values of E_L and θ_L from [5], [6], [12], and [13]:

[61] $x = E_e E_L \sin^2(\theta_L/2)/[E_P(E_e - E_L \cos^2(\theta_L/2))]$

[62] $y = [E_e - E_L \cos^2(\theta_L/2)]/E_e$

Consequently, we shall regard E_e, E_P, x, and y to be the set of known parameters for a given scattering event. The "virtual photon" (or momentum transfer) 4-vector is given in the lab frame by

[63] $\mathfrak{q} = \mathfrak{p}_e - \mathfrak{p}_L = [(E_e - E_L), (-E_L \sin\theta_L), 0, (E_e - E_L \cos\theta_L)]$

where E_L, $\cos\theta_L$, and $\sin\theta_L$ are respectively given in terms of E_e, E_P, x, and y in [18], [20], and [21].

To find the center-of-mass frame, we first rotate the lab frame axes $\hat{x}$, $\hat{z}$ to axes $\hat{v}$, $\hat{t}$ such that transverse ($\hat{t}$) components of $\boldsymbol{q}$ and $\boldsymbol{P}$ cancel:

[64] $\hat{v} \equiv -\cos\eta\,\hat{z} - \sin\eta\,\hat{x}$

[65] $\hat{t} \equiv -\sin\eta\,\hat{z} + \cos\eta\,\hat{x}$

[66] $\boldsymbol{P}\cdot\hat{t} \equiv -\boldsymbol{q}\cdot\hat{t}$

We substitute spatial components of $\mathfrak{q}$, [63], and $\mathfrak{P}$, [58], into [66] in order to find that

[67] $\tan\eta = E_L \sin\theta_L/(E_P + E_L\cos\theta_L - E_e)$
$= 2[xy(1-y)E_eE_P]^{1/2}/[E_P(1-xy) - E_ey]$

[68] $\sin\eta = 2[xy(1-y)E_eE_P]^{1/2}/\{E_P(1-xy) - E_ey]^2 + 4xy(1-y)E_eE_P\}^{1/2}$

[69] $\cos\eta = [E_P(1-xy) - E_ey]/\{[E_P(1-xy) - E_ey]^2 + 4xy(1-y)E_eE_P\}^{1/2}$

We now boost along the $\hat{v}$ axis to a (primed) reference frame moving with velocity u relative to the lab frame such that

[70] $P_v' = -q_v'$

Since

[71] $P_v' = (1-u^2)^{-1/2}[\boldsymbol{P}\cdot\hat{v} - uE_P]$

[72] $q_v' = (1-u^2)^{-1/2}[\boldsymbol{q}\cdot\hat{v} - u(E_e - E_L)]$

we find that

[73] $u = (\boldsymbol{P}\cdot\hat{v} + \boldsymbol{q}\cdot\hat{v})/(E_P + E_e - E_L)$
$= \{[E_P(1-xy) - E_ey]^2 + 4xy(1-y)E_eE_P\}^{1/2}/[E_P(1-xy) + E_ey]$

[74] $\gamma \equiv (1-u^2)^{-1/2} = [E_P(1-xy) + E_ey]/[4y(1-x)E_eE_P]^{1/2}$

Finally, we again rotate axes in the primed frame such that $\boldsymbol{q}'$ $(= -\boldsymbol{P}')$ is in the $+\hat{z}'$ direction (and such that the electron momentum $\boldsymbol{p}_e'$ is in the $(\hat{x}', \hat{z}')$ plane with a positive $\hat{x}'$ component[2]). If we define the rotation angle ϕ such that

[75] $\hat{z}' \equiv -\cos\phi\,\hat{v}' - \sin\phi\,\hat{t}$

[76] $\hat{x}' \equiv -\sin\phi\,\hat{v}' + \cos\phi\,\hat{t}$

then the requirement that $\boldsymbol{P}'\cdot\hat{x}' = 0$ leads to the following expressions for ϕ:

[77] $\tan\phi = 2E_P[x(1-x)(1-y)]^{1/2}/[E_P(1-2x+xy) - E_ey]$

[78] $\sin\phi = 2E_P[x(1-x)(1-y)]^{1/2}/\{[E_P(1-xy) - E_ey]^2 + 4E_eE_Pxy(1-y)\}^{1/2}$

[79] $\cos\phi = [E_P(1-2x+xy) - E_ey]/\{[E_P(1-xy) - E_ey]^2 + 4E_eE_Pxy(1-y)\}^{1/2}$

We summarize the Lorentz transformation to the γ^*P c.m. frame as follows:
Suppose we have a 4-vector $\mathfrak{A} = (A_0, A_x, A_y, A_z)$ in the lab frame, and we wish to find $\mathfrak{A}' = (A_0', A_x', A_y', A_z')$ in the γ^*P c.m. frame. We see that

[80] $A_0' = \gamma(A_0 + u\sin\eta\,A_x + u\cos\eta\,A_z)$

[81] $A_x' = \gamma u\sin\phi\,A_0 + (\gamma\sin\eta\sin\phi + \cos\eta\cos\phi)A_x + (\gamma\cos\eta\sin\phi - \sin\eta\cos\phi)A_z$

[82] $A_y' = A_y$

[83] $A_z' = \gamma u\cos\phi\,A_0 + (\gamma\sin\eta\cos\phi - \cos\eta\sin\phi)A_x + (\gamma\cos\eta\cos\phi + \sin\eta\sin\phi)A_z$

where $\sin\eta$, $\cos\eta$, u, γ, $\sin\phi$, and $\cos\phi$ are given by [68], [69], [73], [74], [78], and [79], respectively.
The inverse transformation between $\mathfrak{A}'$ and $\mathfrak{A}$ is given by

[84] $A_0 = \gamma(A_0' - u\sin\phi\,A_x' - u\cos\phi\,A_z')$

[85] $A_x = -\gamma u\sin\eta\,A_0' + (\gamma\sin\eta\sin\phi + \cos\eta\cos\phi)A_x' + (\gamma\sin\eta\cos\phi - \cos\eta\sin\phi)A_z'$

[86] $A_y = A_y'$

[87] $A_z = -\gamma u\cos\eta\,A_0' + (\gamma\cos\eta\sin\phi - \sin\eta\cos\phi)A_x' + (\gamma\cos\eta\cos\phi + \sin\eta\sin\phi)A_z'$

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[2]Consequently, the positive x' axis represents azimuthal zero, and a single hadron current jet would be expected at an azimuthal angle of 180°.

TABLE 1. Jet energies E_J (GeV) and angles θ_J (degrees) for collisions between 1000 GeV protons and 10 GeV electrons. x and y are assumed to be obtained from outgoing lepton kinematics, as described in the text. The following cases have been considered: (I) no parton or lepton masses ($m_i = m_f = m_L = 0$); (II) scattering off a heavy quark ($m_i = m_f = 5$ GeV, $m_L = 0$); (III) production of a heavy lepton ($m_L = 5$ GeV, $m_i = m_f = 0$); (IV) production of a very heavy lepton ($m_L = 25$ GeV, $m_i = m_f = 0$); and (V) production of a very heavy parton ($m_f = 25$ GeV, $m_i = m_L = 0$). Blank entries are kinematically forbidden at x and y values indicated

		Case I		Case II		Case III		Case IV		Case V	
x	y	E_J	θ_J	E_J	θ_J	E_J	θ_J	E_J	θ_J	E_J	θ_J
0.05	0.05	48.0	168.3	48.74	168.4	47.4	168.2	32.4	165.7	360.5	178.4
0.05	0.30	38.0	147.4	38.74	147.8	37.4	147.1	22.4	137.0	90.1	166.3
0.05	0.80	18.0	96.4	18.74	97.9	17.4	94.5	—	—	37.5	140.3
0.30	0.05	285.5	175.2	298.6	175.4	284.8	175.2	269.9	175.1	598.0	177.7
0.30	0.30	213.0	166.4	213.6	166.4	212.3	166.4	197.4	165.8	265.1	169.0
0.30	0.80	68.0	139.9	68.6	140.2	67.3	139.7	52.4	134.0	87.5	148.5
0.80	0.05	760.3	177.1	761.1	177.1	759.6	177.1	744.9	177.0	—	—
0.80	0.30	562.8	171.6	563.6	171.6	562.1	171.6	547.4	171.5	615.1	175.7
0.80	0.80	167.8	154.8	168.6	154.9	167.1	154.7	152.4	153.5	187.5	157.4

Using [80]–[83], we can determine the 4-momenta of the scattering particles in the γ^*P c.m. frame. For example, the lab frame proton 4-vector $\mathfrak{P}$, [58], transforms to

[88] $$\mathfrak{P}' = [E_e E_P y/(1-x)]^{1/2}(1, 0, 0, -1)$$

and the lab frame 4-vector $\mathfrak{q}$, [63], transforms to

[89] $$\mathfrak{q}' = [E_e E_P y/(1-x)]^{1/2}(1-2x, 0, 0, 1)$$

Also, the leptonic 4-momenta in the γ^*P c.m. frame are given by

[90] $$\mathfrak{p}_e' = [E_e E_P y/(1-x)]^{1/2}[(1-xy)/y]\left\{1, 2\frac{[x(1-x)(1-y)]^{1/2}}{1-xy}, 0, \frac{(1-2x+xy)}{1-xy}\right\}$$

[91] $$\mathfrak{p}_L' = \mathfrak{p}_e' - \mathfrak{q}'$$

$$= [E_e E_P y/(1-x)]^{1/2}[(1-y+xy)/y]\left\{1, 2\frac{[x(1-x)(1-y)]^{1/2}}{1-y+xy}, 0, \frac{1-y+xy-2x}{1-y+xy}\right\}$$

Recall from Sect. III that the incident parton 4-momentum $\mathfrak{p}_i$ is a fraction $\xi = x$ of the proton 4-momentum $\mathfrak{P}$ if the partons are on-mass-shell; consequently,

[92] $$\mathfrak{p}_i' = \xi\mathfrak{P}' = [E_e E_P y/(1-x)]^{1/2}(x, 0, 0, -x)$$

The current jet J has the energy and momentum of the outgoing parton,

[93] $$\mathfrak{p}_J' = \mathfrak{p}_i' + \mathfrak{q}' = [E_e E_P y/(1-x)]^{1/2}[1-x, 0, 0, (1-x)]$$

and the target fragments T have energy and momentum

[94] $$\mathfrak{p}_T' = \mathfrak{P}' - \mathfrak{p}_i' = [E_e E_P y/(1-x)][1-x, 0, 0, -(1-x)]$$

VI. Conclusions

In this paper we have attempted to develop an error-free description of the kinematics of electron–proton collisions in both the lab and the γ^*P c.m. frames. Particular attention has been paid to parton and lepton mass effects, as it will be important to be able to distinguish empirically between "nuisance" masses, which one would like to neglect in order to test the nature of scaling violation, and those masses which would indicate the presence of new physics.

In Table 1, lab frame values are tabulated for E_J and θ_J for a variety of choices for m_i, m_f, and m_L and assuming beam energies appropriate for CHEER. Given an empirical range of values for x ($\equiv Q^2/2m_P\nu$) and y ($\equiv \nu/\nu_{max}$) between 0.05 and 0.8, we see that jet kinematics are virtually the same for massless processes as they are for scattering off massive bottom quarks, or even for producing a 5 GeV massive lepton! Target mass corrections (m_P) are completely negligible for the range of x and y considered. Consequently, a machine such as CHEER is insensitive to *expected* mass-effects and should be capable of testing QCD predictions of scaling violation without mass-effect ambiguities. *Unexpected* mass-effects are seen to carry their own kinematic signatures. For example, production of an extremely heavy lepton ($m_L = 25$ GeV) decreases the jet energy and angle (as defined in Fig. 1) from values obtained in the massless case for the same x and y; moreover, production of an extremely heavy parton ($m_f = 25$ GeV) *increases* jet energy and angle.

Finally, we note that our kinematic scaling variable, [49], is much less sensitive to mass effects than the "operator product" expression, [50]. This is because mass-effects will not occur in either case unless Q^2 is sufficiently small to be comparable to parton masses, but for such values of Q^2, $Q^2 - q_1{}^2 \simeq Q^2 y \ll Q^2$.

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