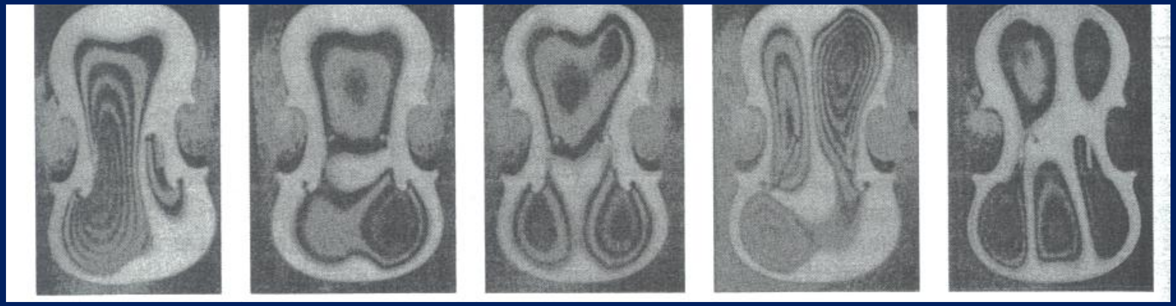
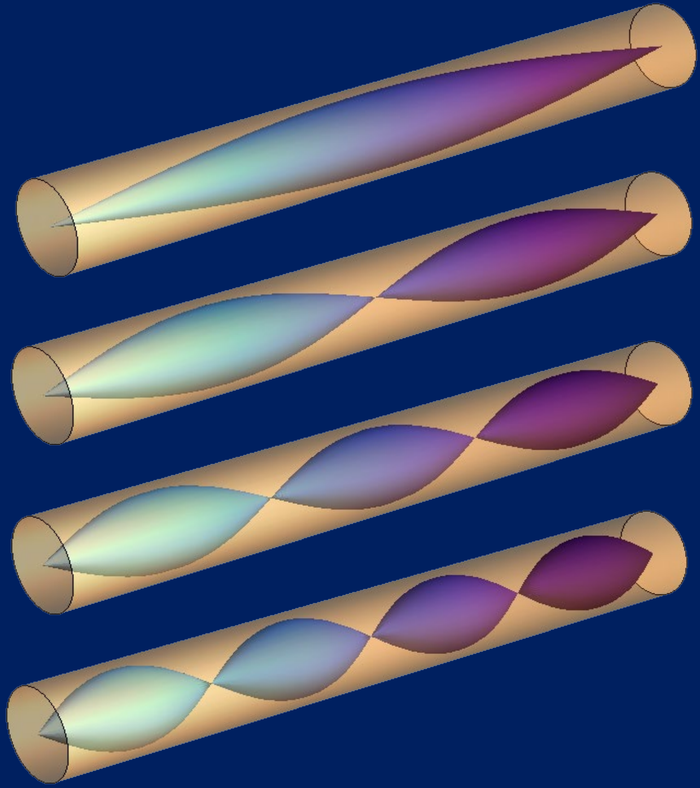


The Physics of Music



Christine A. Aidala

University of Michigan

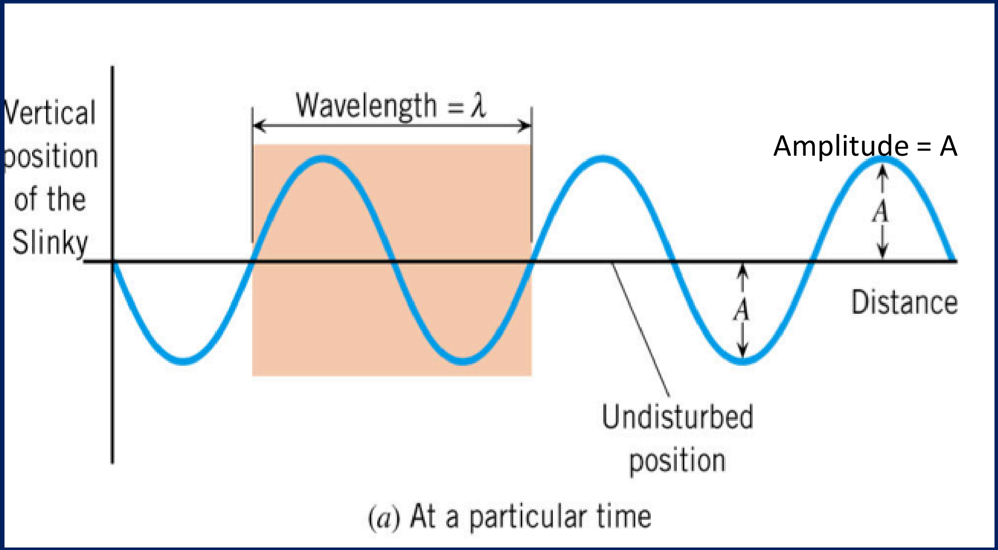
Saturday Morning Physics

October 23, 2021

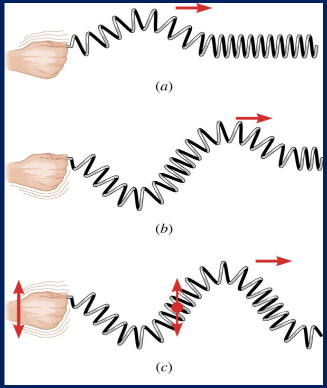
Waves

$$v = \lambda f$$

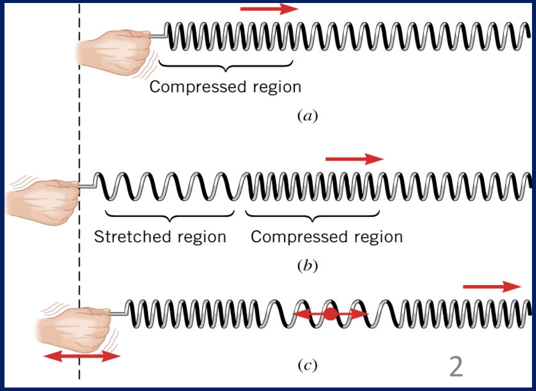
speed = wavelength x frequency



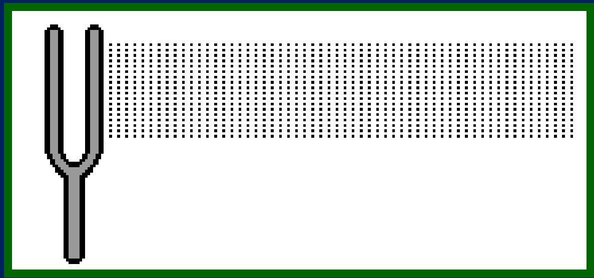
- Transverse waves
  - A disturbance *perpendicular* to the *direction of travel*
  - E.g. electromagnetic waves (radio, microwaves, visible light, x-rays, gamma rays)

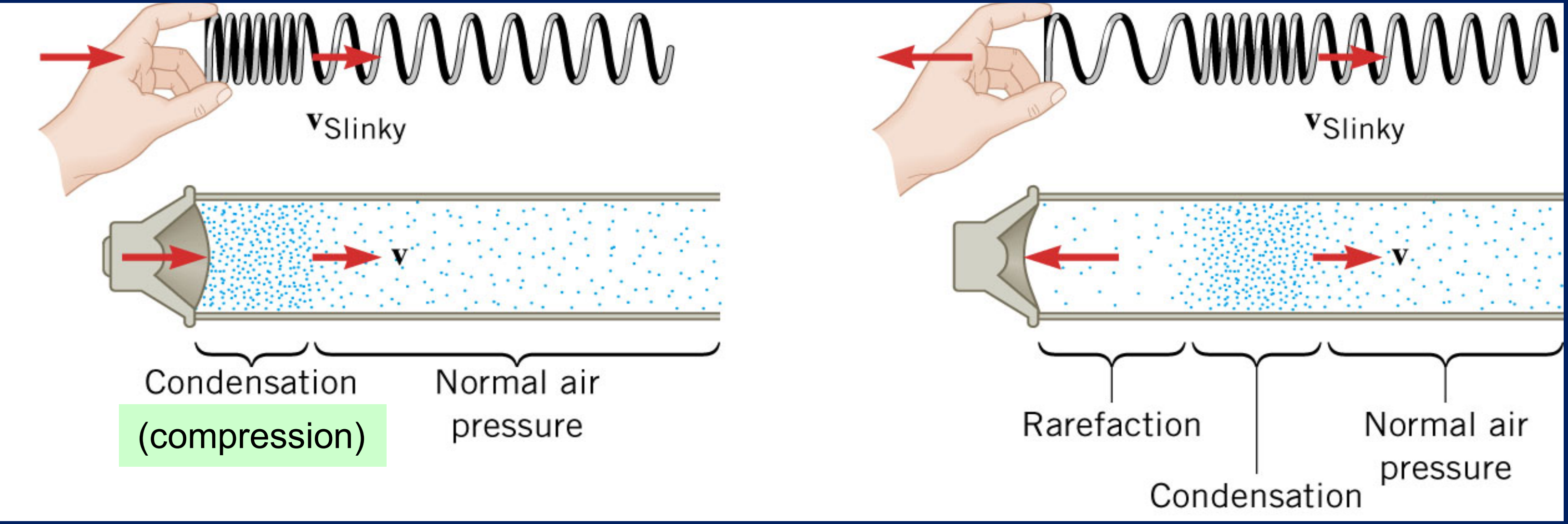


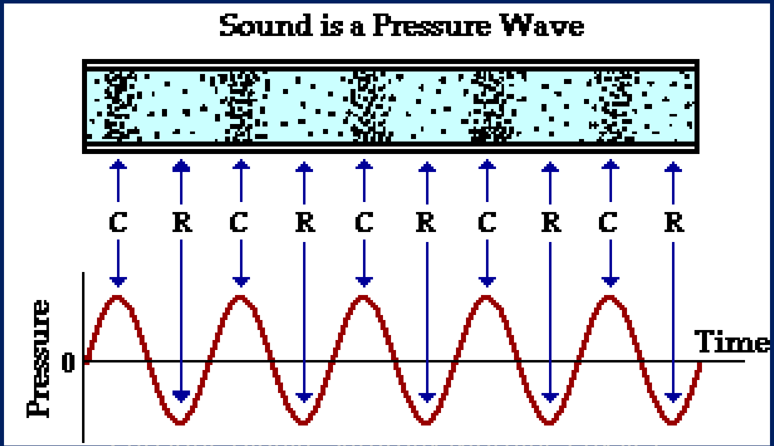
- Longitudinal waves
  - A disturbance *parallel* to the *direction of travel*
  - Sound waves are longitudinal air compression waves



Sound is a Pressure Wave





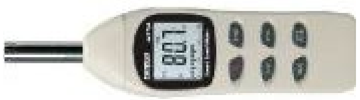
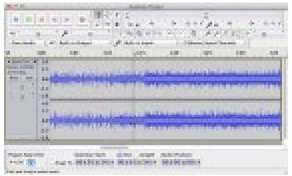


Characterizing Musical Tones

- When we listen to a note, we can describe its properties by *loudness, pitch* and *timbre*
- We can physically measure *intensity*, *frequency* and *waveform*

Perception of sound

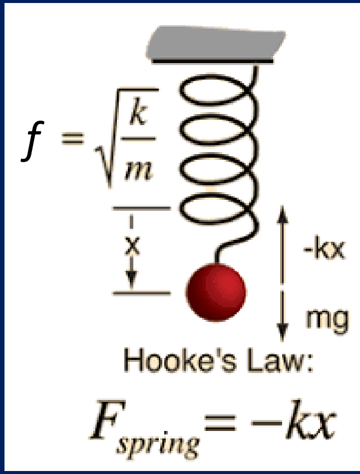
Physical properties

Perception	**Physical**	
Loudness how loud of soft a note sounds	Intensity measured W/m^2 or dB, proportional to amplitude^2	
Pitch high or low notes on a musical scale	Frequency cycles per second (Hz)	
Timbre tone color or "quality" of the sound	Waveform measured shape of the wave	

Describing Oscillatory Motion

- Oscillations (vibrations) occur because of
  - Restoring force: Acts to move the object *back* to its *equilibrium* position
  - Inertia: Acts to maintain the motion in its current direction and speed
- Frequency of oscillations depends on strength of force and amount of inertia

$$f \propto \sqrt{\frac{\text{Restoring force}}{\text{Inertia}}}$$



  - Stronger restoring force produces higher frequency (faster oscillation)
  - More inertia gives lower frequency (slower oscillation)

Vibrating Stretched String

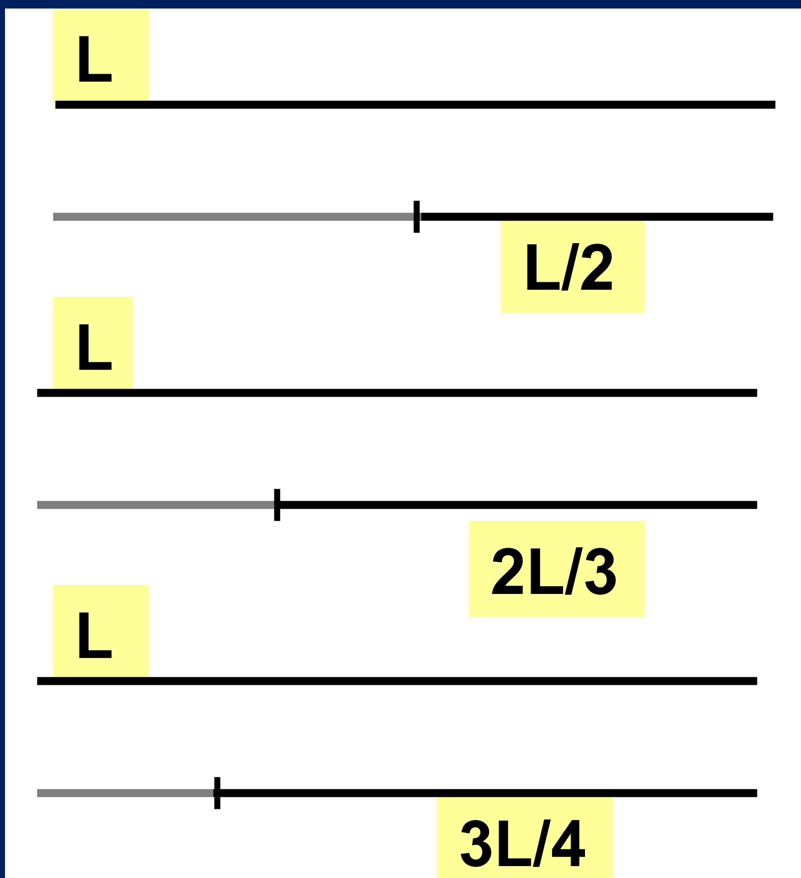
- Frequency depends on
  - String tension T
  - Mass per unit length μ
  - Total length of string L
- Properties
  - Greater tension → higher frequency
  - Greater mass → lower frequency
  - Longer string → lower frequency
- For same string stopped at different lengths:

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

$$f_2 / f_1 = L_1 / L_2$$

Intervals and Ratios

- Ancient Greeks: Tone intervals we hear depend on *ratios* of string lengths plucked
  - Some intervals seem "natural" to us – "consonance"



"octave"
ratio: L/(L/2) = 2/1 or "**2 to 1**"

"perfect fifth"
ratio L/(2L/3) = 3/2 or "**3 to 2**"

"perfect fourth"
ratio L/(3L/4) = 4/3 or "**4 to 3**"

Natural Modes of Vibration of a String

- Both ends are fixed: force “nodes” at the endpoints
  - Nodes are points that stay at equilibrium
  - Where nodes are depends on the mode (except ends)
- Fit integer number of half-waves $\lambda/2$ into length L

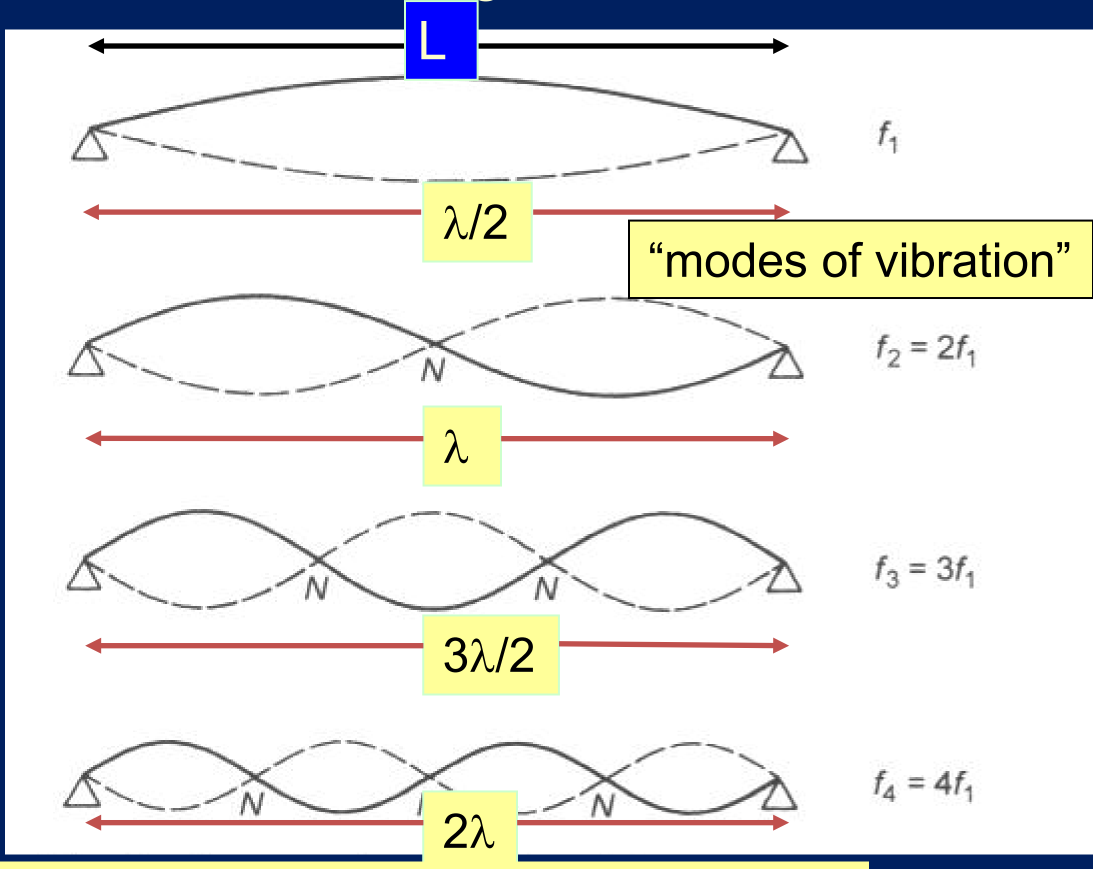
Standing waves on string demo

$L = \lambda/2 \Rightarrow \lambda_1 = 2L$

$L = \lambda \Rightarrow \lambda_2 = L$

$L = (3/2)\lambda \Rightarrow \lambda_3 = (2/3)L$

$L = 2\lambda \Rightarrow \lambda_4 = (1/2)L$

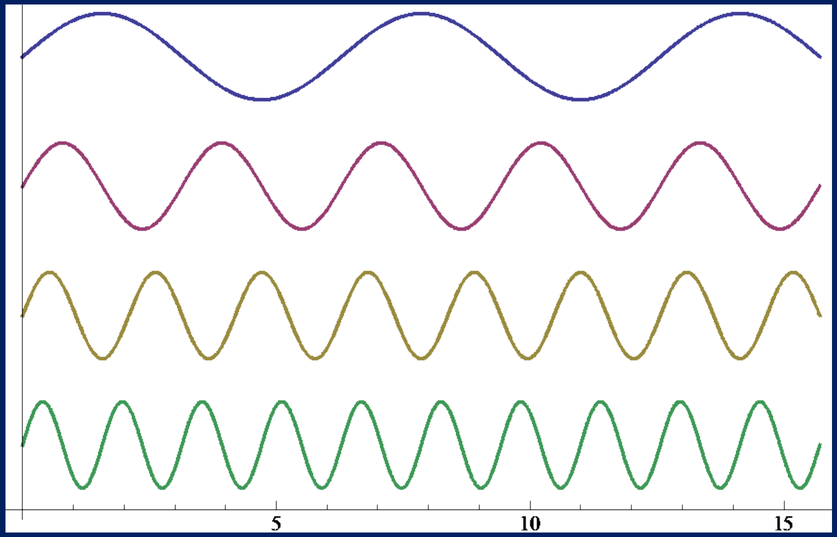


In general for n^th mode: $\lambda_n = 2L / n \qquad n = 1, 2, 3,\ldots$

Harmonics and the Harmonic Series

- The higher-frequency natural modes of string vibration are known as harmonics in math/science
  - Music: "overtones" or "partials"
- A set of frequencies $f, 2f, 3f, 4f, 5f$, ... is known as a harmonic series, and *f* is known as the *fundamental frequency*

f	1st harmonic	fundamental	1st partial
$2f$	2nd harmonic	1st overtone	2nd partial
$3f$	3rd harmonic	2nd overtone	3rd partial
$4f$	4th harmonic	3rd overtone	4th partial



The Harmonic Series as Notes

- A harmonic series can start at any fundamental frequency
  - For example, start at A_2 = 110 Hz

f_1	110 Hz	A_2
f_2	220 Hz	A_3
f_3	330 Hz	E_4
f_4	440 Hz	A_4
f_5	550 Hz	$C\sharp_5$
f_6	660 Hz	E_5
f_7	770 Hz	?
f_8	880 Hz	A_5

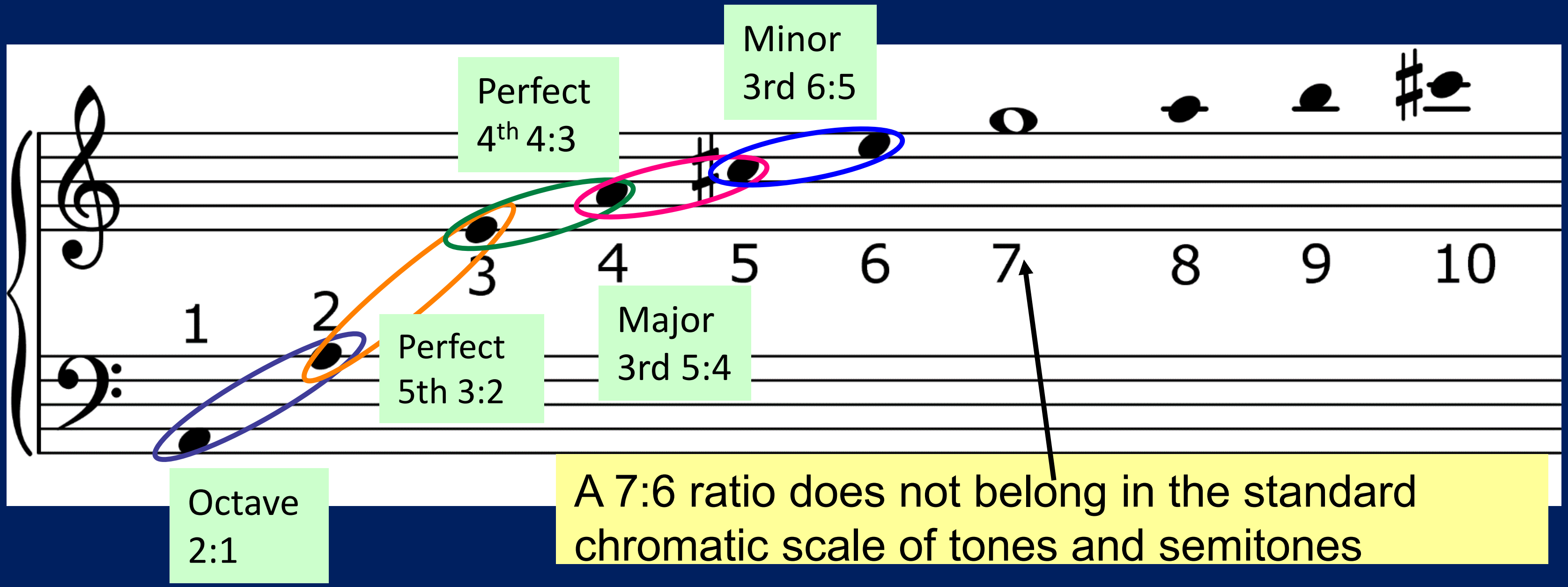
Octaves are based on powers of two (1, 2, 4, 8,…)

Most (but not all) frequencies match notes on the 12-tone chromatic scale

In between $G\flat_5$ and G_5

The Harmonic Series on A_2

- The first few terms in the harmonic series starting on A_2

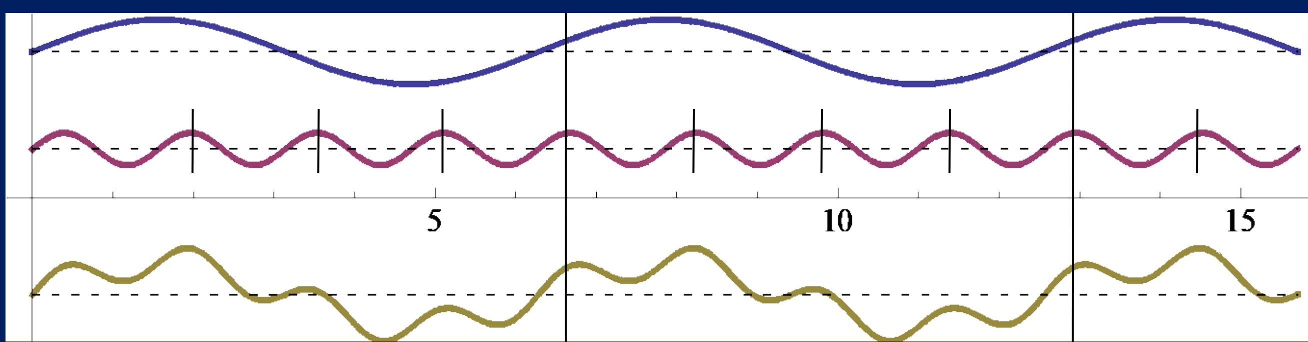


- Musical harmony seems to be related to the harmonic series
  - Consonant intervals are related to frequencies in a harmonic series

Strings Vibrate in Multiple Natural Modes Simultaneously!

- In general, a string vibrates *simultaneously* with many frequencies in the harmonic series ($f, 2f, 3f, 4f, \ldots$)
  - Higher harmonics typically have smaller amplitudes
- Simplified example of what vibrating at just two natural mode frequencies simultaneously looks like:

f and $4f$



Frequency f

Frequency $4f$

Combined motion

"Ripples" of frequency $4f$ on top of a wave of frequency f

Timbre and Waveform

- Different instruments sound different, even when playing the same note!
  - Timbre: the “tone quality”
  - Due to the relative strengths of the various harmonics
  - Complex sound waveforms

- *Can understand complex waveforms by pulling them apart into simpler components*

Complex Waveforms and Fourier Analysis

- Any periodic waveform of fundamental frequency f can be expressed as the sum of sine waves of frequencies $f, 2f, 3f, 4f, \ldots$
  - Terms in the harmonic series!

Periodic waveform Fundamental f	analyze $\rightarrow$ $\leftarrow$ synthesize	Sum of sine waves Frequencies f, $2f$, $3f$, $4f$,…

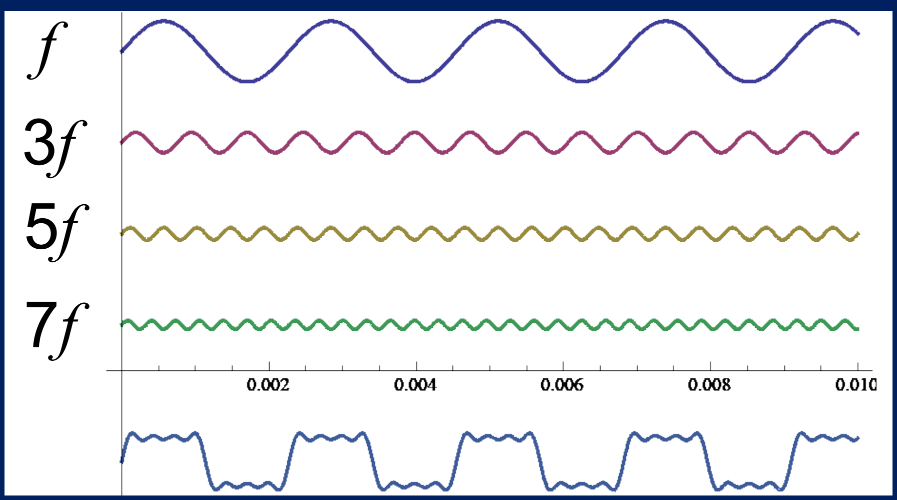
- Need specific amplitudes for the various contributing frequencies to get a specific waveform
- Fourier analysis forms much of the basis for analyzing musical tones:

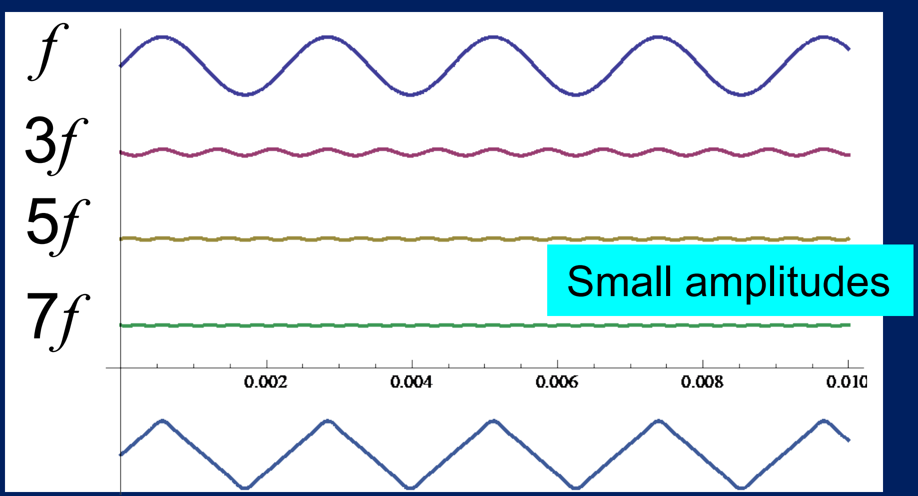
> The fundamental determines the pitch, and the relative strengths of the higher harmonics determine the timbre

Fourier Example: Building Square and Triangle Waves

- We can build a square wave out of odd harmonics
- The more harmonics we add, the closer we get to a true square wave

- The triangle wave also has only odd harmonics, but with different amplitudes
- The triangle wave is closer to a sine wave than a square wave
  - Does not need as much "adjustment"



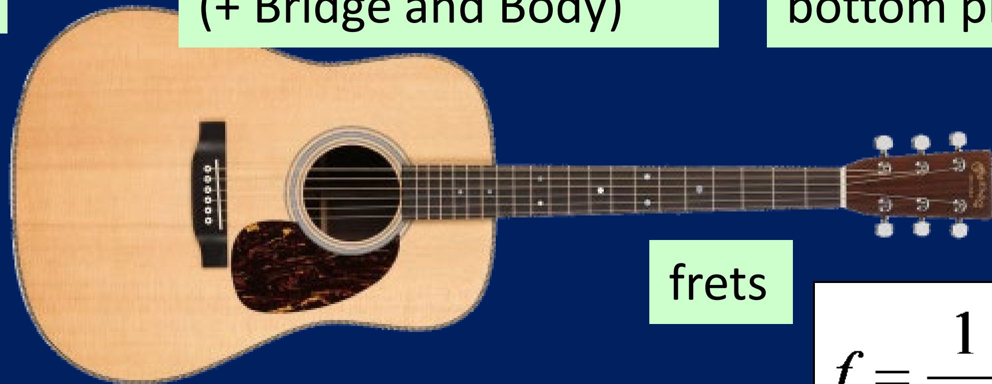


Sound from a Guitar

- Acoustic guitar

Exciter → Vibrator → Radiator

Exciter	Vibrator	Radiator
Pluck the string	String (+ Bridge and Body)	Body (top and bottom plates)



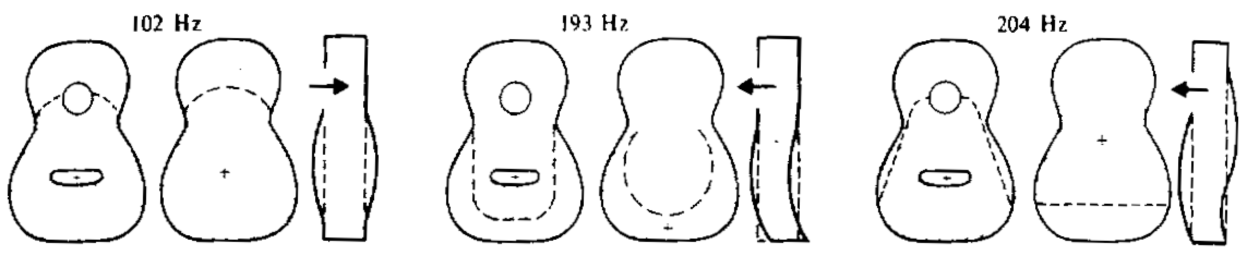
frets

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

- The strings themselves produce very little sound
  - Ineffective at pushing air around
- What we hear comes from the guitar body

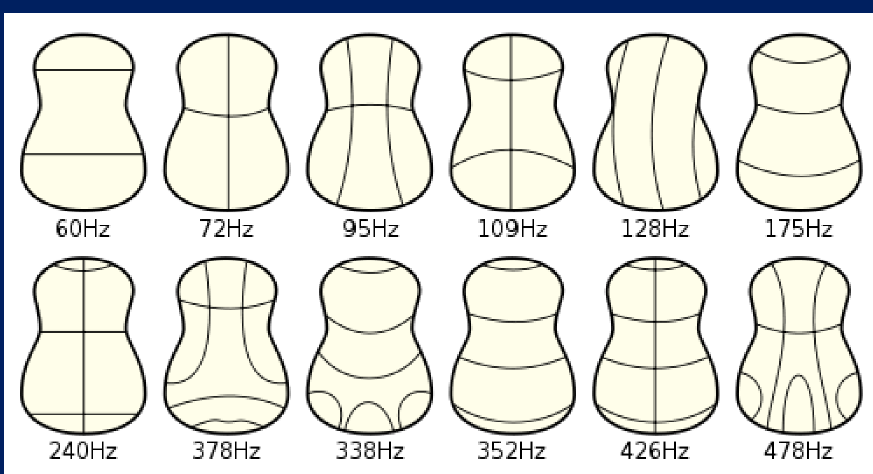
Body Resonances

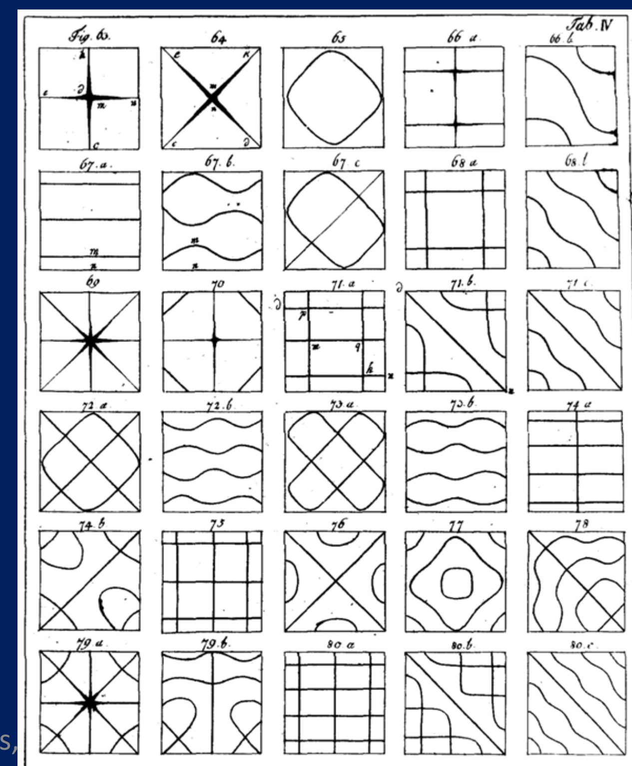
- The sound of an acoustic guitar is different from that of an electric guitar
  – Or banjo, mandolin, plucked violin, viola, . . .
- Important coupling between natural modes of the vibrating strings and modes of the body
- Lowest three modes of a Martin D-28 folk guitar



Natural Modes of Different Shaped Plates: Chladni Plates

- Sprinkle powder on a plate and vibrate it
- When the driving frequency matches a natural mode frequency (resonance!) the powder accumulates at nodal lines
  - Recall: nodes stay at equilibrium
- Can see resulting complex patterns of two-dimensional modes
- Used to study and design violin or guitar plates

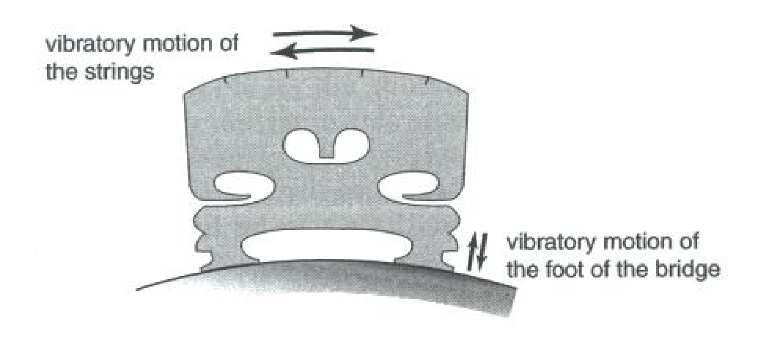


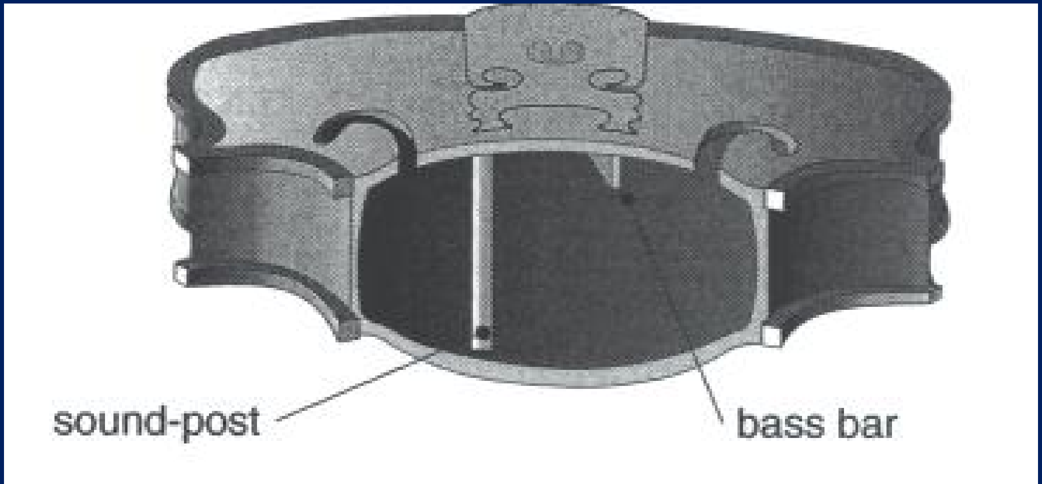


Chladni plate demo

Violin Acoustics

- Violin sound comes from:
  - How the bow excites the string vibrations
  - Coupling of the string vibrations to the body through the bridge
  - Behavior of the body as a resonator – the top plate, back plate, body cavity, f-holes, etc.



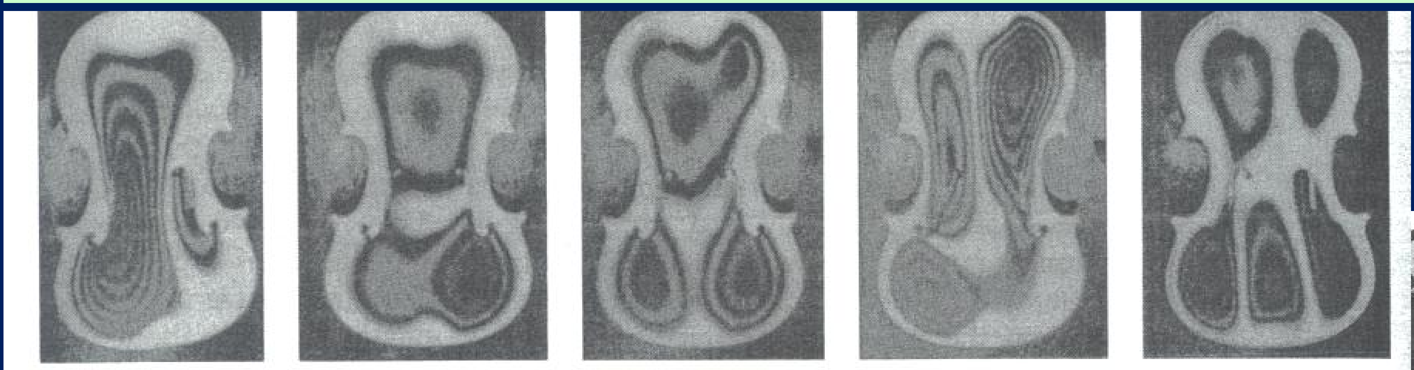


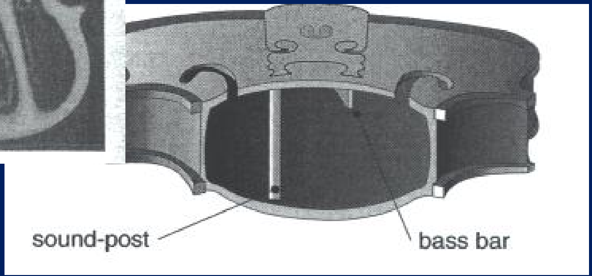
The body has inside supports
 - Makes the violin asymmetrical!

Body Resonances

- The pitch of a note is determined by the string – not by the natural modes of the body
- Why do we care about the body modes?
  - Because of resonance – if the frequency of the fundamental or an overtone matches the frequency of a natural mode of vibration of the body, that harmonic gets strengthened relative to the others
- Different stringed instruments have different body modes → different timbres

Violin top plate vibrations at various frequencies – not very symmetric!

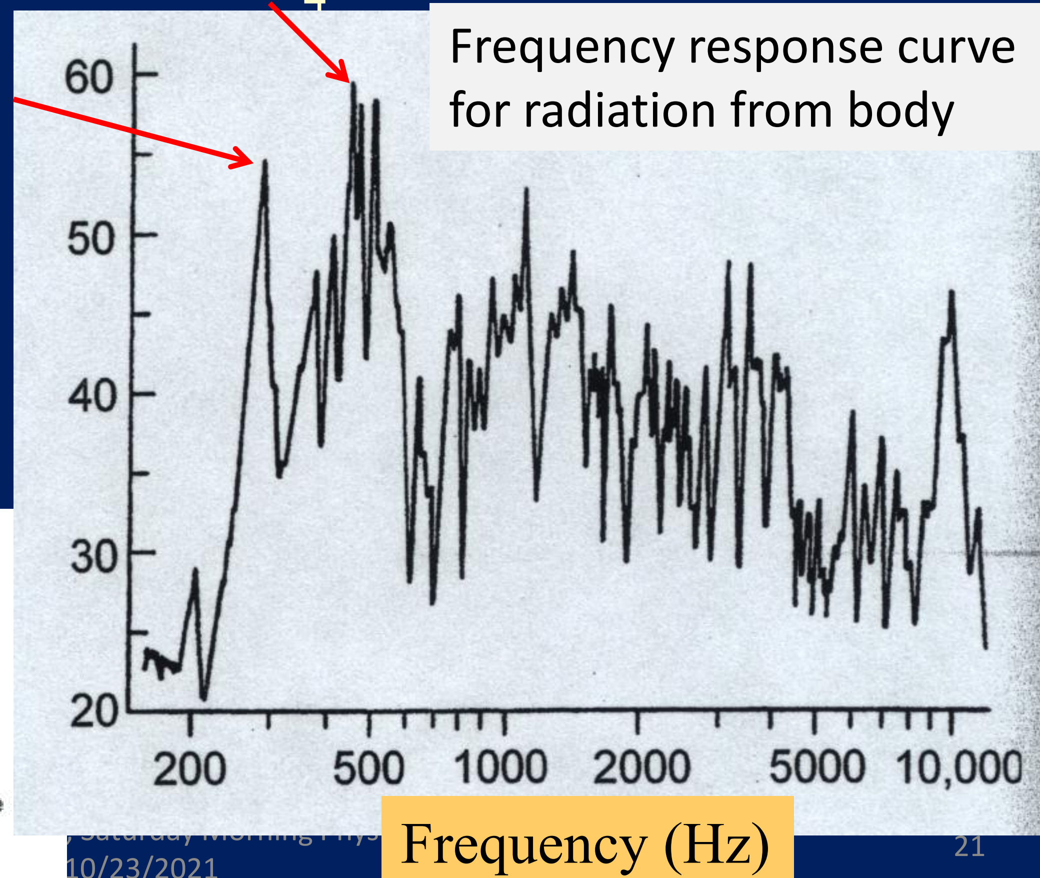


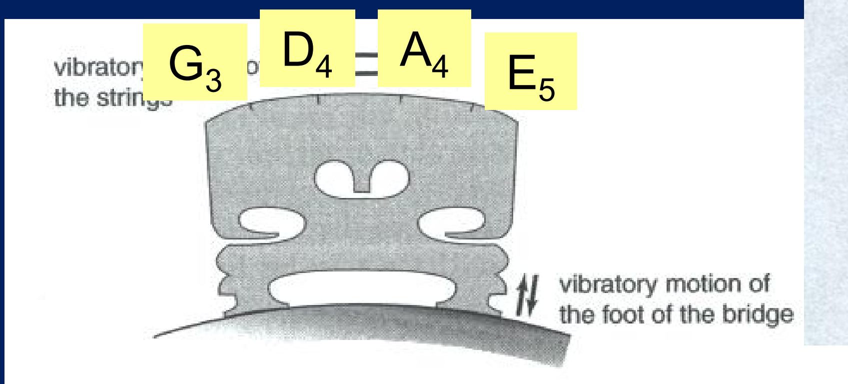


Violin Body Resonances

- The violin body has two main bands of resonance
  - Main wood resonance – near A_4
  - Resonance of air in the body – near D_4

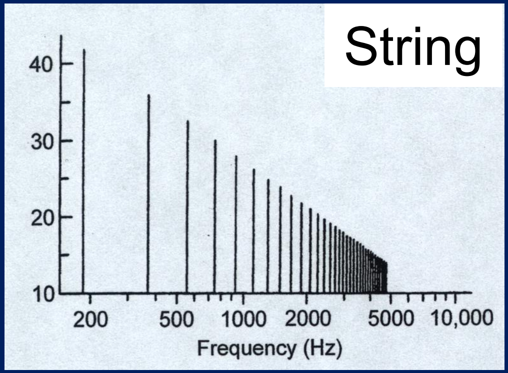
Donald Hall, *Musical Acoustics*

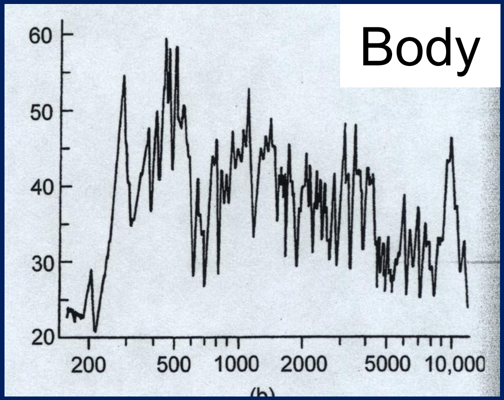




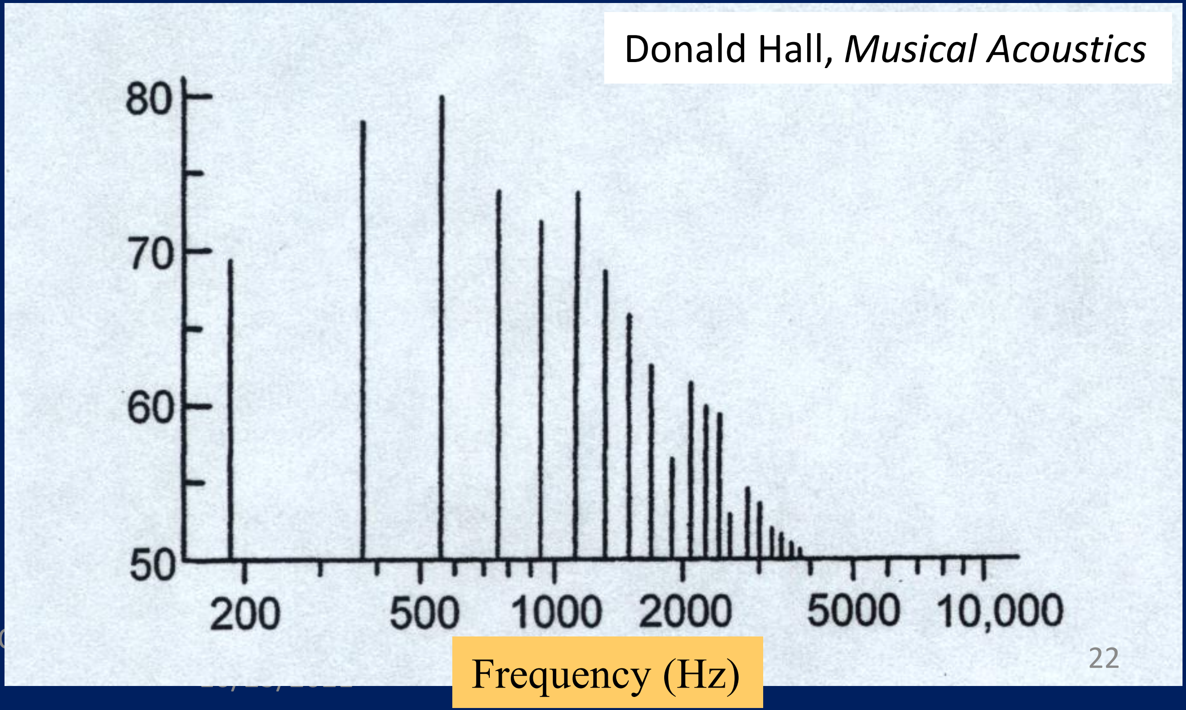
Radiated Violin Sound

- The sound we hear from the violin is a combination of the vibration recipe of the string and the frequency response of the body





For the open G string
(196 Hz, lowest note on violin)



Winds

- Blow into a pipe!
  - Excite vibrations in a column of air
- Classify wind instruments based on how the sound is excited

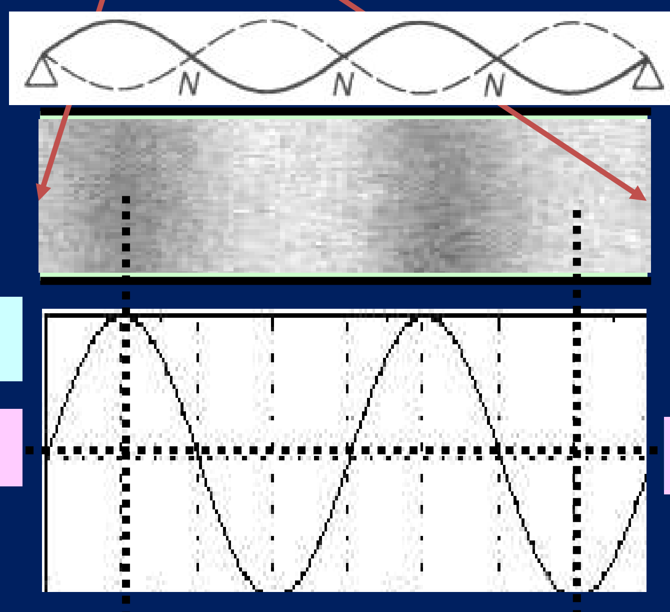
Air jets	Reeds	Brass
		

- Produces sound of a definite pitch – related to the length of the tube

Air Pipe with Open Ends

- Any open end of an air pipe must have a pressure node
  - Because the pressure at the open end must be the same as the outside pressure
  - This is always a node

Pressure nodes of an air column are just like displacement nodes of a string



Pressure graph

outside pressure

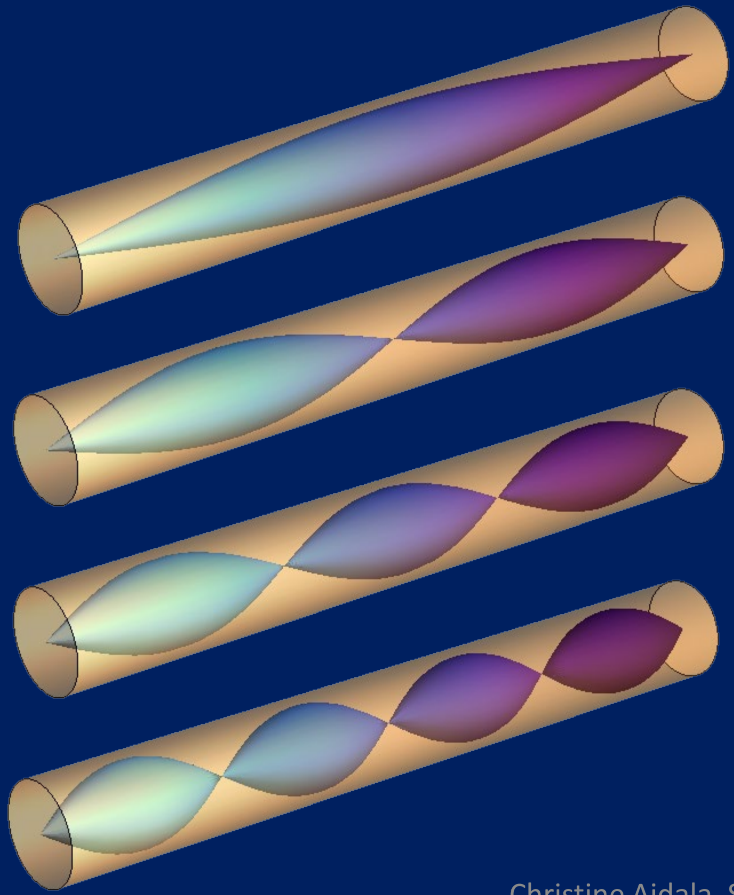
outside pressure

pressure max

pressure min

Natural Modes of Open Pipes

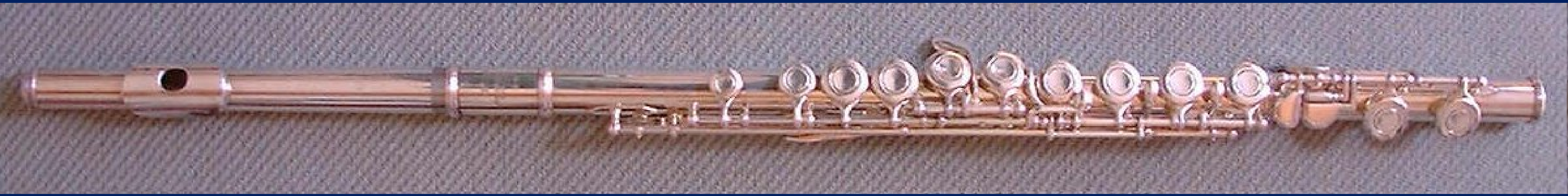
- Open pipe modes form a harmonic series



n = 1	f_1	Fundamental
n = 2	$f_2=2f_1$	2nd harmonic
n = 3	$f_3=3f_1$	3nd harmonic
n = 4	$f_4=4f_1$	4th harmonic

Ruben's tube demo

The Flute

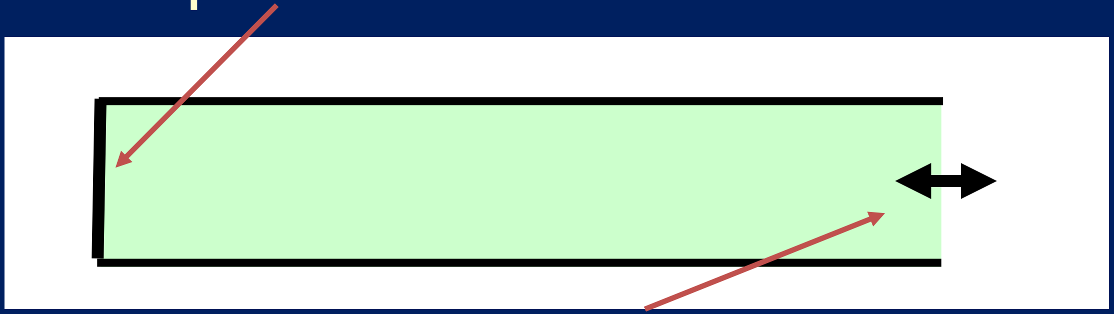


- Blowing across the mouthpiece excites the vibrations
- Effective length of the pipe determines the pitch
  - Pressure will equilibrate with atmospheric pressure at first open tone hole
- 12 tone holes for one octave of the chromatic scale
  - Overblow to excite higher harmonics for the second and third octaves
- The piccolo is essentially a half-size flute
  - So it plays one octave up

Flute demo

Air Pipe with One Closed End

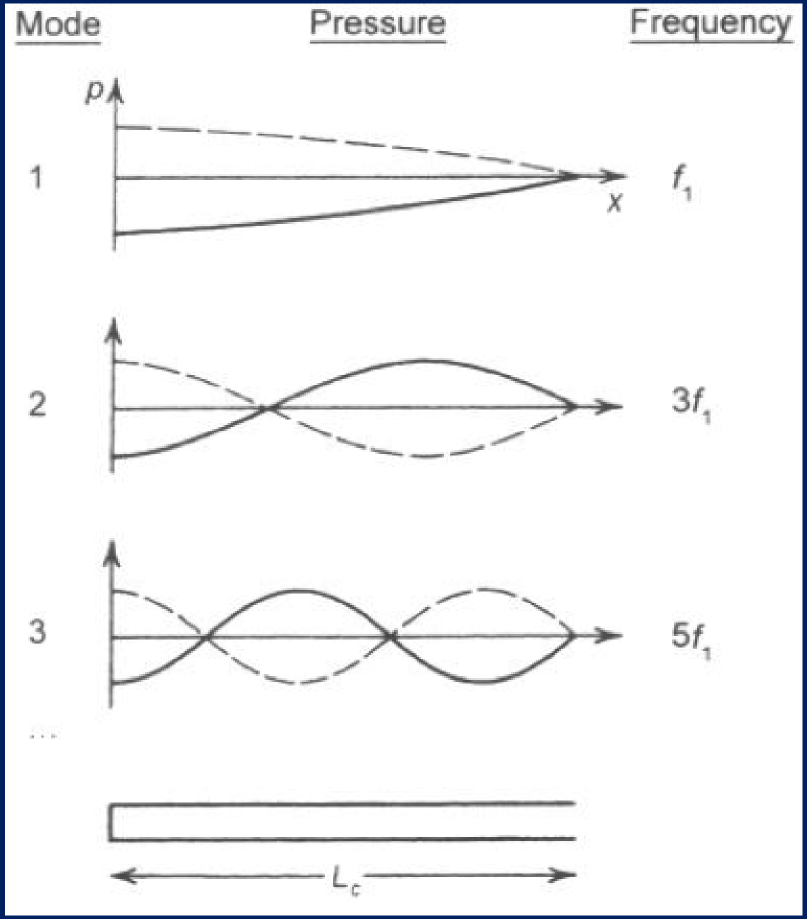
- We can also excite air vibrations using a reed
  - The vibrating reed acts like a valve to open and close the flow of air
- At the closed end of an air column, pressure can build up



  - Pressure at open end equilibrates with atmospheric pressure—a pressure node
- This is called a “closed pipe”, even though one end is open

Closed Cylindrical Pipes - Clarinet

- Because pressure can build up at the closed end, closed cylinder natural modes are different from open pipes
- The fundamental corresponds to one *quarter cycle* of a wave, rather than one half
- Same length pipe will play an *octave lower* than an open one
- The clarinet overblows to $3f$ (octave + fifth) and $5f$ (two octaves + major third)
  - Only odd harmonics: accounts for the hollow timbre of the clarinet's low notes



Donald Hall, *Musical Acoustics*

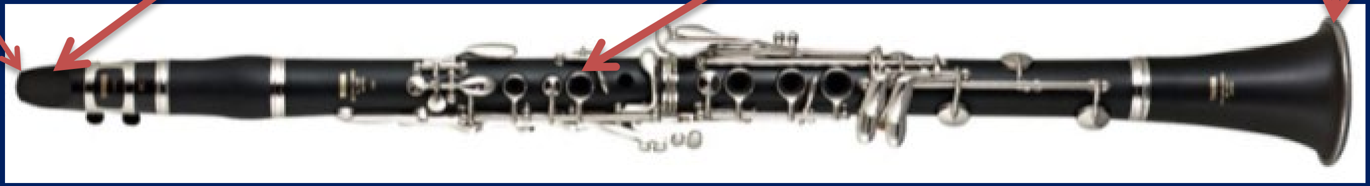
Sound from a Clarinet

- Clarinet (single reed instrument)

Exciter → Vibrator → Radiator

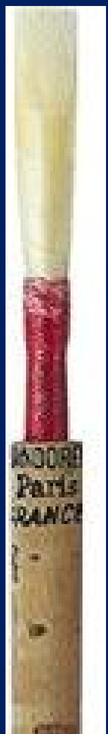
Reed | Body (cylindrical pipe) | Bell

Blow

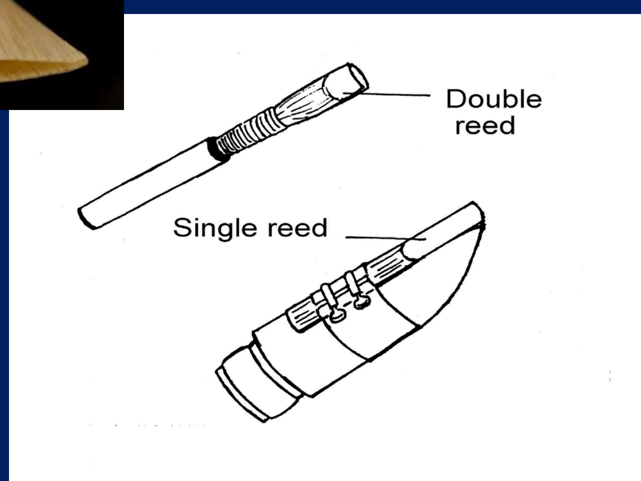


Clarinet demo

Single and Double Reeds



- The reed woodwind instruments can have either single or double reed mouthpieces
  - Acoustical properties not so different
- What matters instead is a cylindrical vs. conical bore of the pipe!
  - Cylindrical: Clarinet
  - Conical: Saxophone, oboe, bassoon

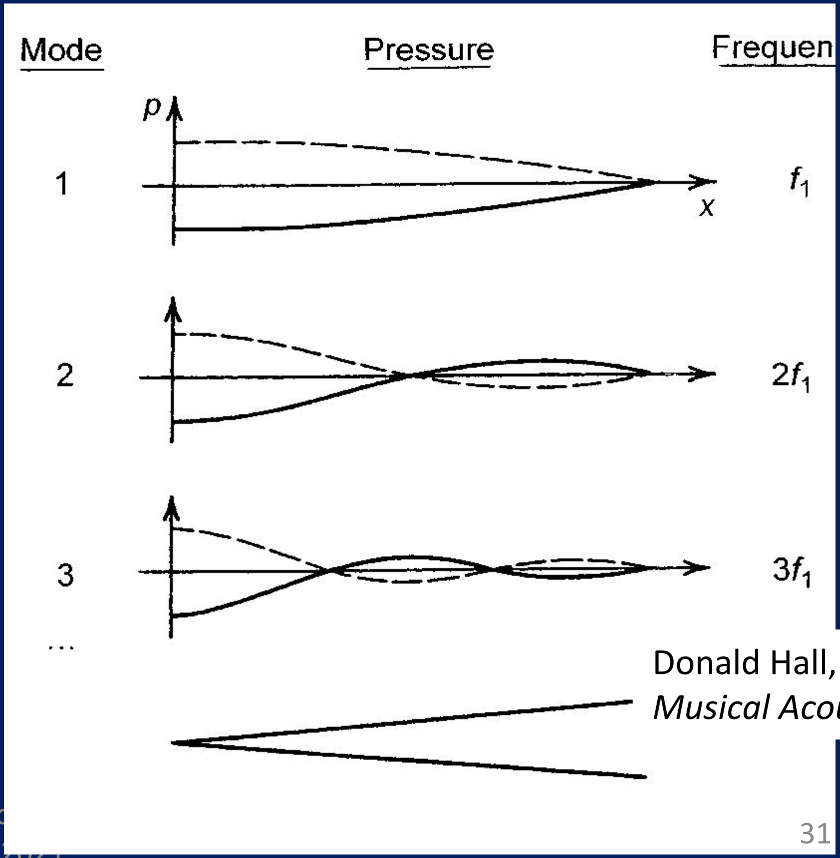


Natural Modes of Cones

- The natural modes of a cone form a harmonic series
  - Not like a cylindrical pipe with closed end
  - Like an open pipe (on both ends), even though it has a reed on one end

As the cone gets wider, not as much pressure can build up in the standing wave, and the positions of the pressure nodes get shifted with respect to those of a cylindrical pipe

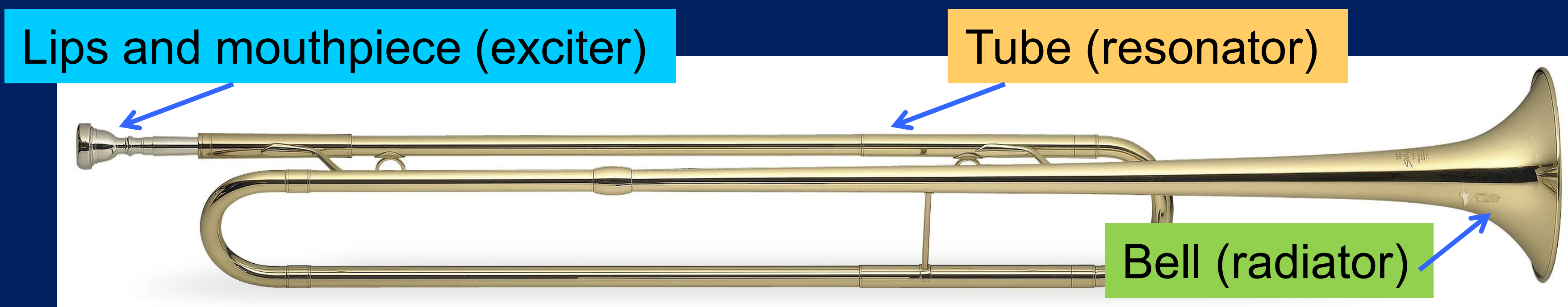
End up with both even and odd harmonics: $f, 2f, 3f, 4f, \ldots$



Donald Hall, *Musical Acoustics*

Brass Instruments

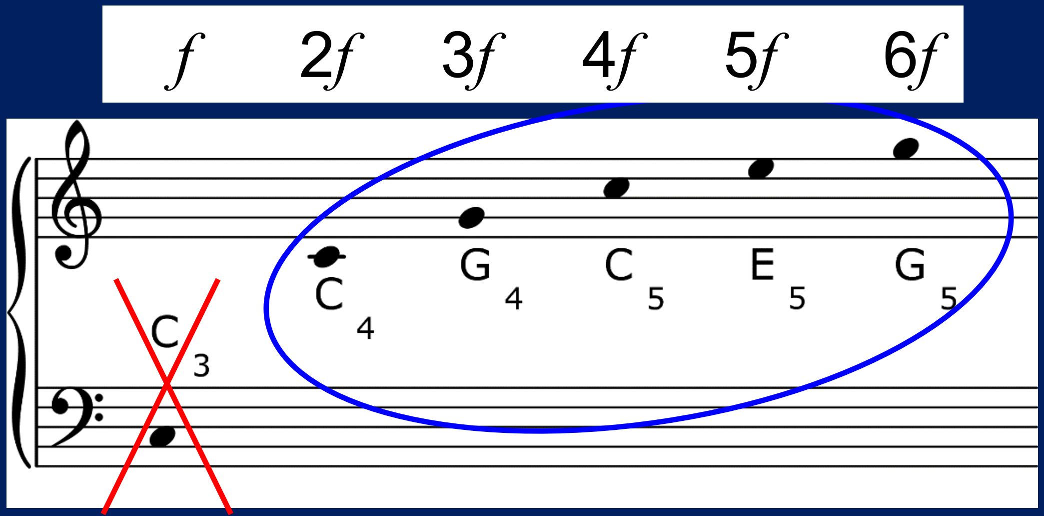
- Lip buzzing (and mouthpiece buzzing) is what excites vibrations and sound waves

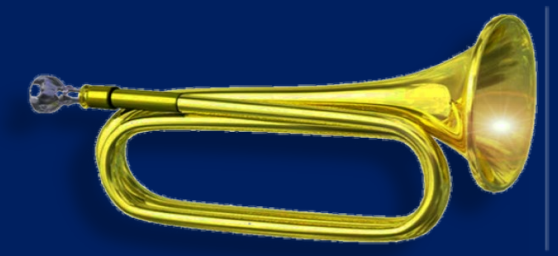


Brass mouthpiece demo

Harmonic Series for Brass

Example: Bugle notes





Get a harmonic series with both even and odd harmonics for all brass --

<u>but can be very hard to get the fundamental</u>

The Trombone

- Play more than just bugle calls?
- Need more notes!

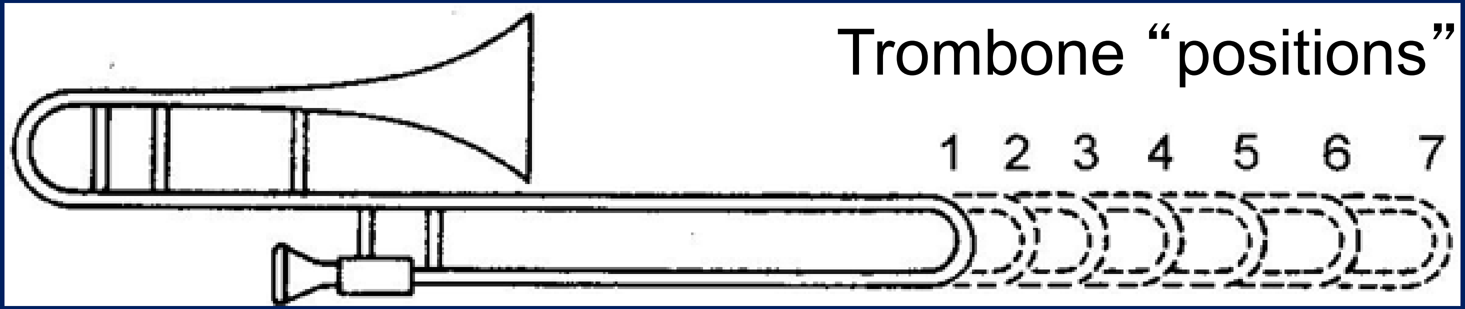
  ⇒ Change the length of the pipe
- The slide trombone does this in a clever manner

  – Two ways to change the pitch:
    play higher harmonics or adjust the slide

Trombone demo

Trombone Slide

- The closed position plays B♭$_2$
- Use the trombone slide to decrease the pitch
  - Go down from B♭$_2$
  - Or down from higher mode notes

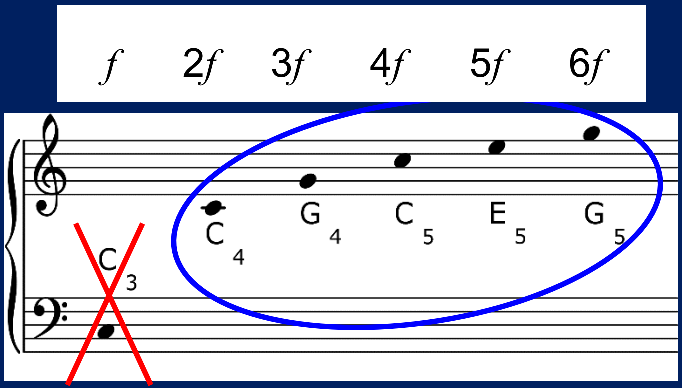


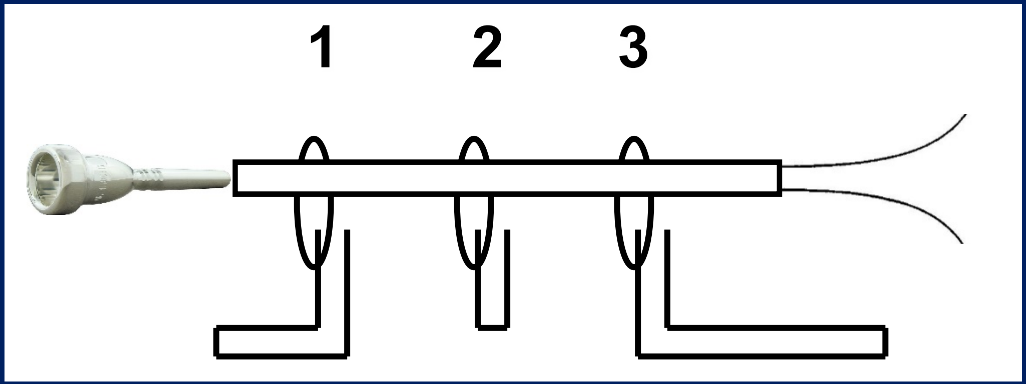
- Each position → lower by a semitone (half step)
- Each semitone shift is <u>relative to the immediately preceding length</u>, not the beginning length

⇒ Actual amount of length shift must steadily increase

Brass Instruments with Valves

- If we want to fill in the notes between G_4 and C_4, we need to add six new pitches
- This can be done using three valves





s^2 s s^3

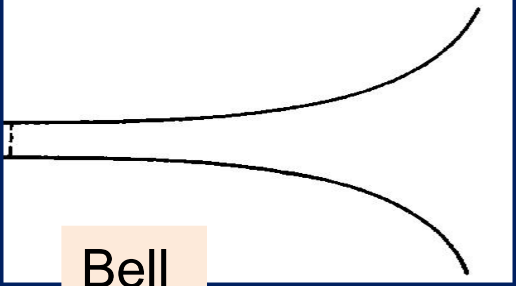
s = semitone

Euphonium demo

Effect of Brass Instrument Bells

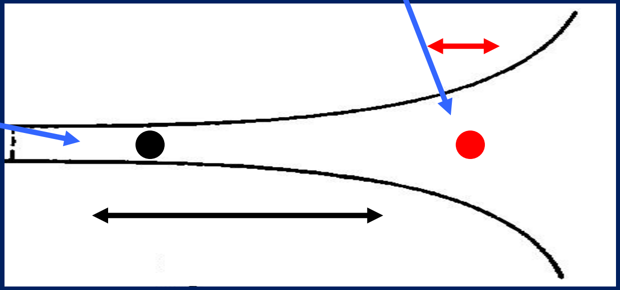
Mouthpiece

Tubing

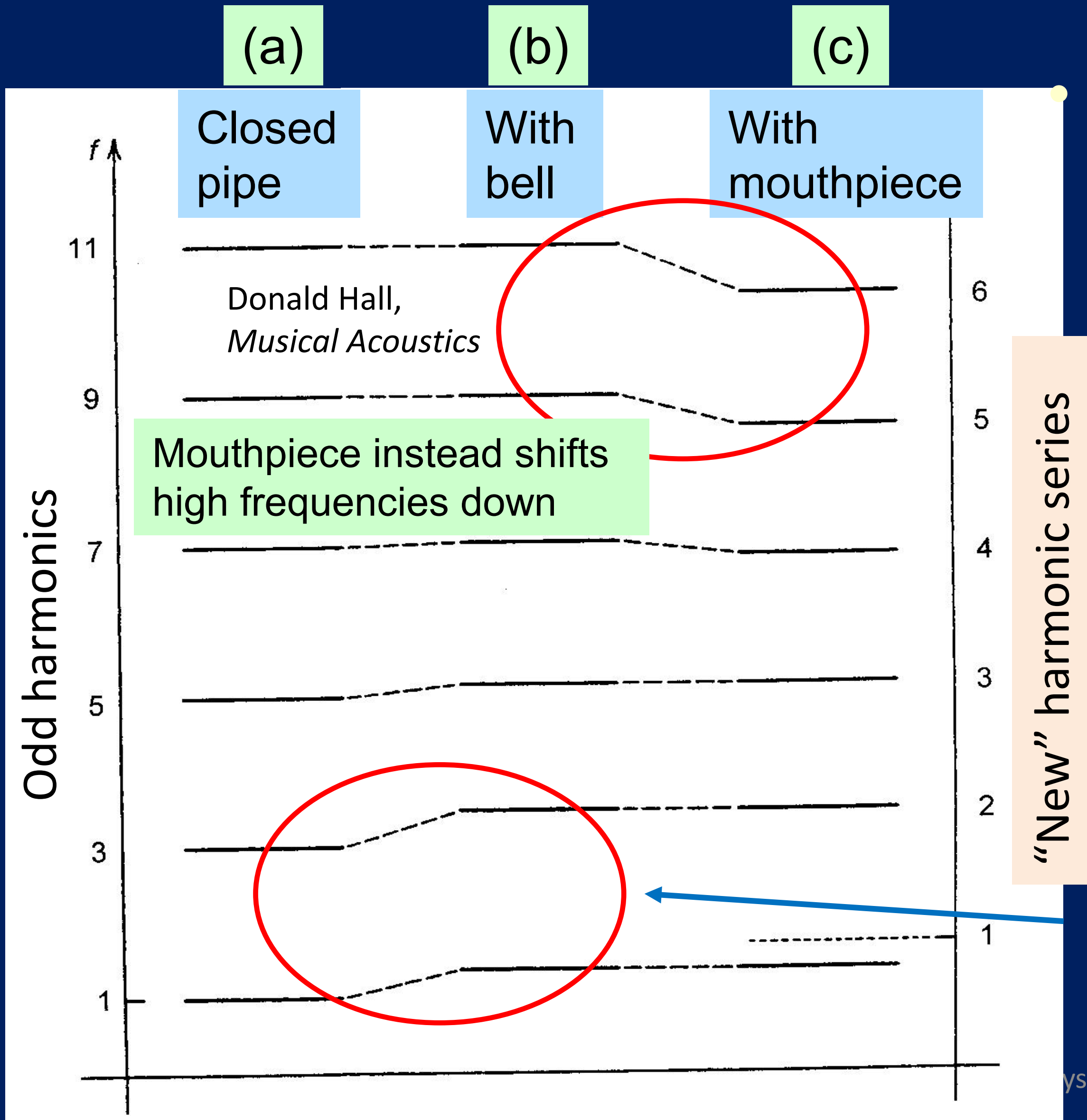


Bell

- Bell: changes cross-sectional area gently
- Small λ: curvature seems gentle to the wave – pressure node near the open end of the bell
- Large λ: curvature seems abrupt to the wave – pressure node far from the open end of the bell
- So where the pressure equilibrates depends on the pitch!



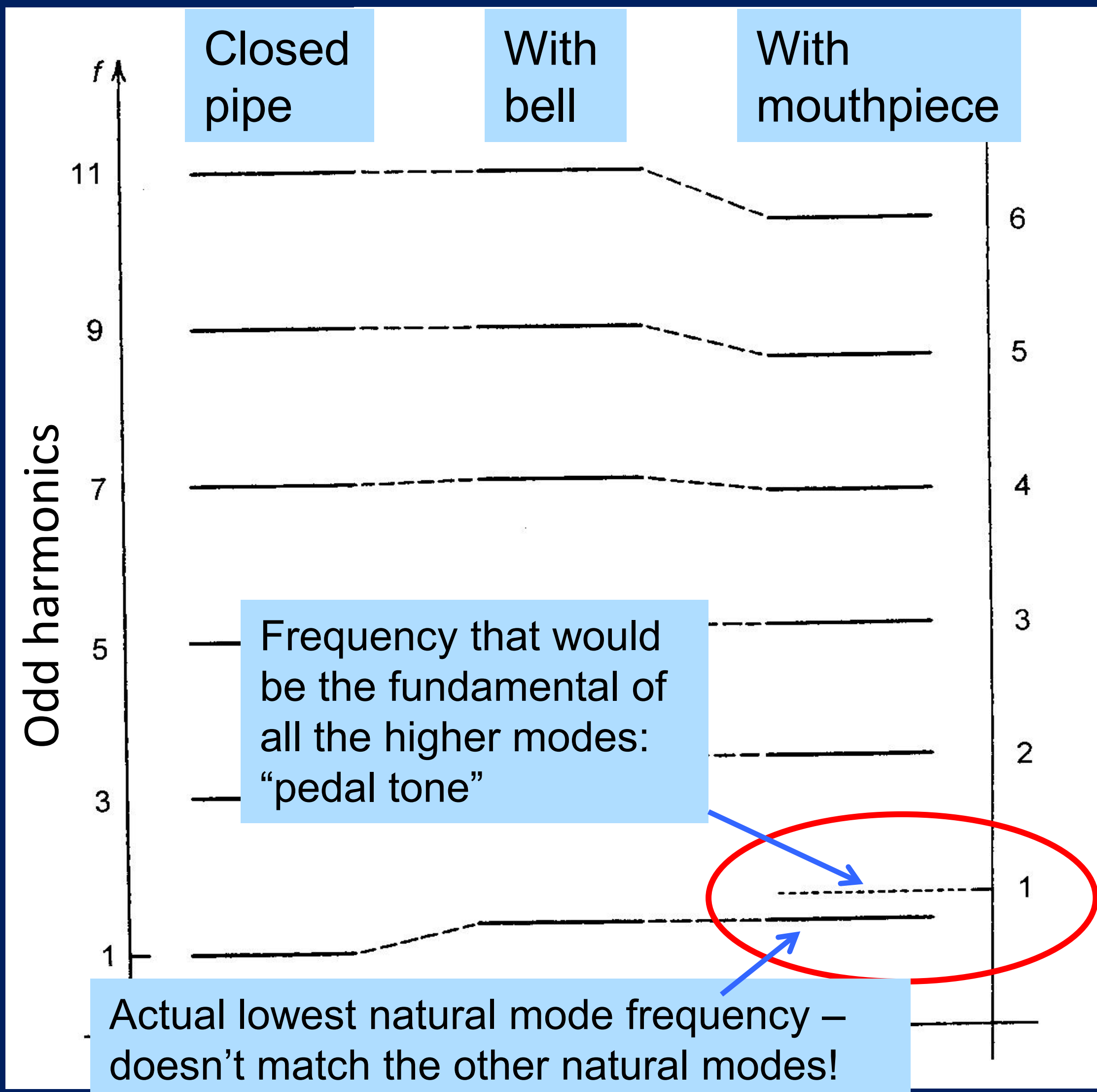
Modification of Mode Frequencies



- Adding a bell and mouthpiece modifies the natural modes of a closed pipe
  a) Cylindrical tube closed at one end
  b) Replace part of tube with bell
  c) Replace opposite end part with mouthpiece

Long wavelengths (low frequencies) effectively "see" a shorter pipe, so their natural mode frequencies get shifted up

Almost Get a New, Full Harmonic Series



Remember:
Sustained periodic wave must be built of components of a harmonic series:
nf, where *n* = 1, 2, 3, …
(Fourier theorem)

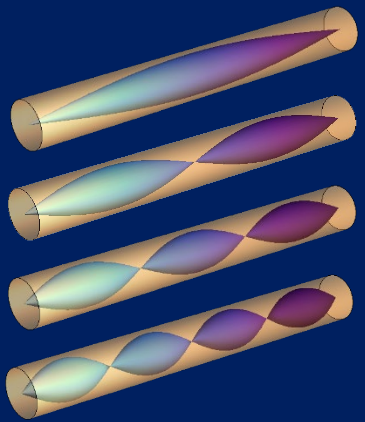
Lowest mode frequency here is wrong for a harmonic series – too low

Makes fundamental very hard to play

Called a “pedal tone”

Pedal tone demo

Summary



- Music is produced via natural modes of vibration – of strings, air columns
  - Also bars and membranes for percussion
- The same note on different instruments sounds different because of the different relative strengths of the higher harmonics
  - Affected by e.g. natural modes of the body for string instruments, how vibrations are excited, open vs. closed and cylindrical vs. conical pipes

I hope a greater understanding of the physics underlying music brings you a deeper appreciation of the musical arts!

Extra

The Speed of Sound

- All waves have a speed associated with them
  - The speed of sound is about 344 m/s (770 mph) at 20° C
  - But depends on temperature

For temperature T in Celsius:

speed (in m/s) = 344 + 0.6 x (T – 20)

  - This can mess up the pitch of organ pipes and wind instruments

Typical Sound Pressures

- Typical sound waves have pressure amplitudes of about 0.01 N/m^2 (quiet) to 1 N/m^2 (loud)
- This means the pressure changes for a loud sound correspond to about
  - 101,326 N/m^2 compressions
  - 101,324 N/m^2 rarefactions
- A fractional change of 1 part in 10^5
  - Quiet sounds may have a fractional change of 1 part in 10^7 or more
- Our ears are really remarkable in that we can detect such small pressure fluctuations

Typical Sound Power

- Compared to a 100 W light bulb, sound power is very small
  - Typically in the μW to mW range
- Maximum sound power

Instrument	Maximum sound power
Bass drum	20 W
Trombone	5 W
Trumpet	300 mW
Flute	50 mW

- Most instruments are at most <u>a few percent efficient</u> in converting input energy to sound energy

Typical Sound Wavelengths

- We can calculate the wavelength of sound waves using $v = \lambda f$
  - the speed of sound in dry air at 20° C is v = 344 m/s

Frequency	Wavelength
27.5 Hz (A_0)	12.5 m
440 Hz (A_4 or concert A)	78 cm
4186 Hz (C_8)	8.2 cm
20 kHz	1.7 cm

Human scale

The size of people, furniture, windows, etc. is important for sound propagation

Typical Sound Intensity Levels

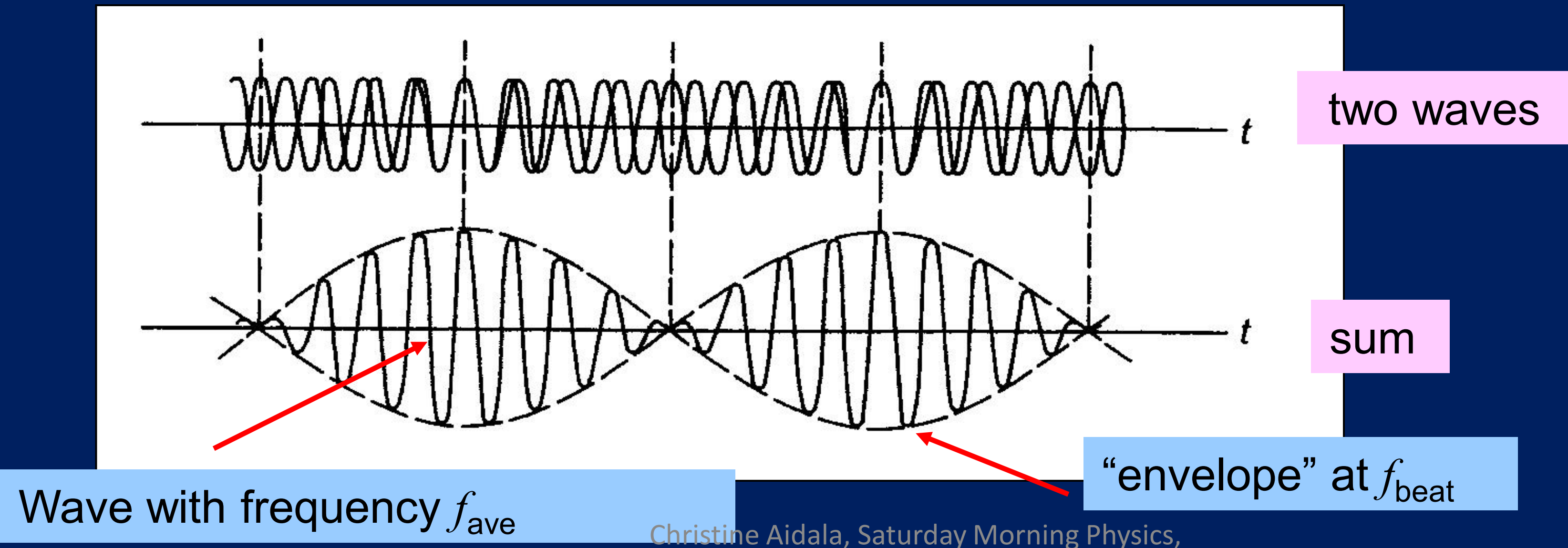
Sound Source	Sound Level (dB)	$I(W/m^2)$	Reaction
Jet engine at 10 m	150	10^3	Unbearable
	140		
	130		
SST takeoff at 500 m	120	1	Painful
Amplified rock music	110		
Machine shop	100		
Subway train	90	10^{-3}	
Factory	80		
City traffic	70		Musically useful
Quiet conversation	60	10^{-6}	
Quiet auto interior	50		
Library	40		
Empty auditorium	30	10^{-9}	
Whisper at 1 m	20		
Falling pin	10		
	0	10^{-12}	Inaudible

Donald Hall, *Musical Acoustics*

Each 10 dB corresponds to a *factor of 10* in intensity

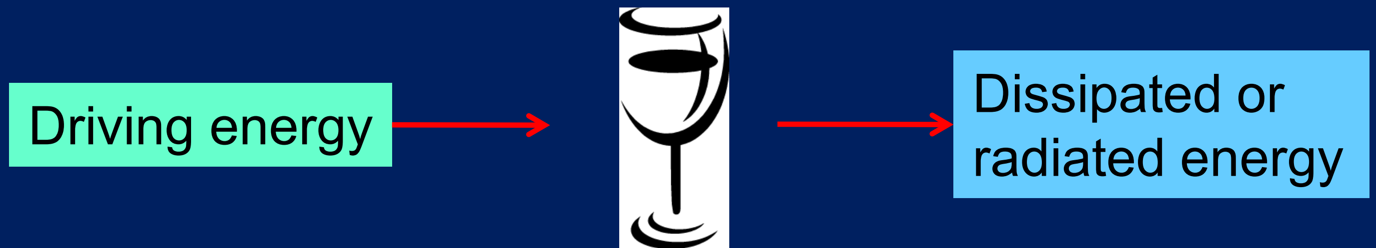
Beats

- The result of adding two waves of nearly equal frequencies f_1 and f_2
  - The resulting wave sounds like a pure tone with frequency $f_{\text{ave}} = \frac{1}{2}(f_1 + f_2)$
  - The sound pulsates at the beat frequency $f_{\text{beat}} = |f_1 - f_2|$

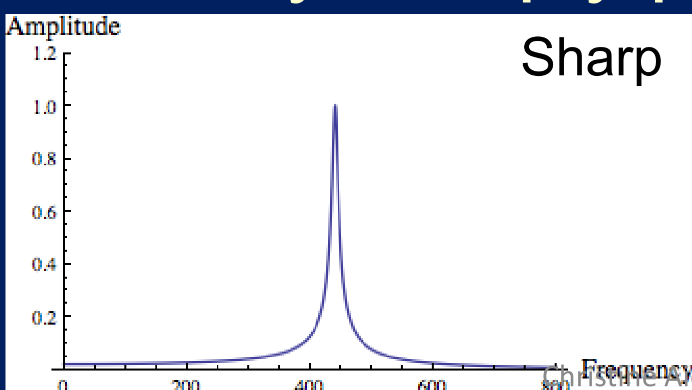


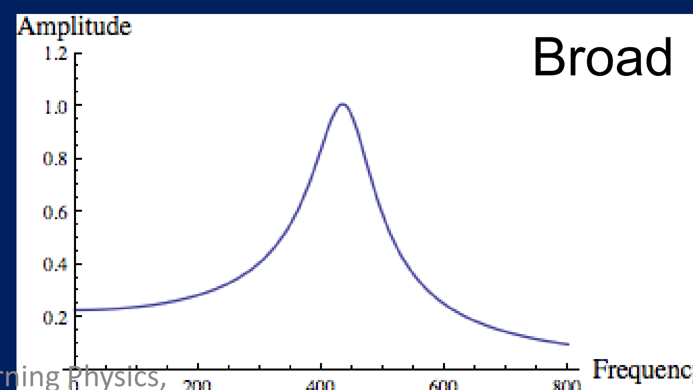
Resonance

- Resonance occurs when we drive a system at one of its natural modes

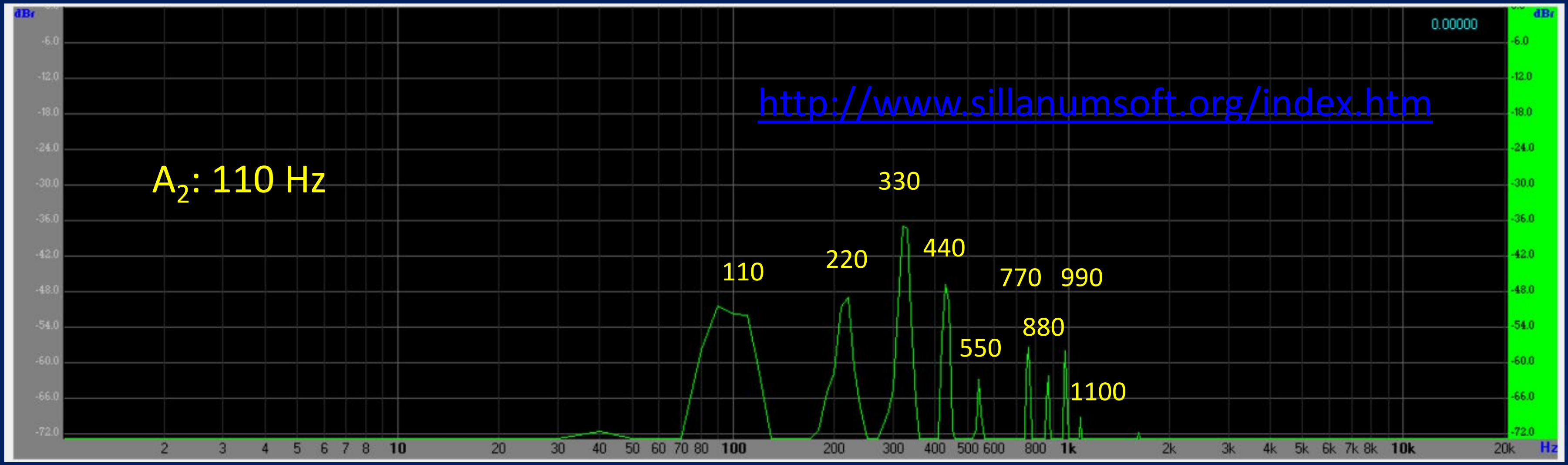


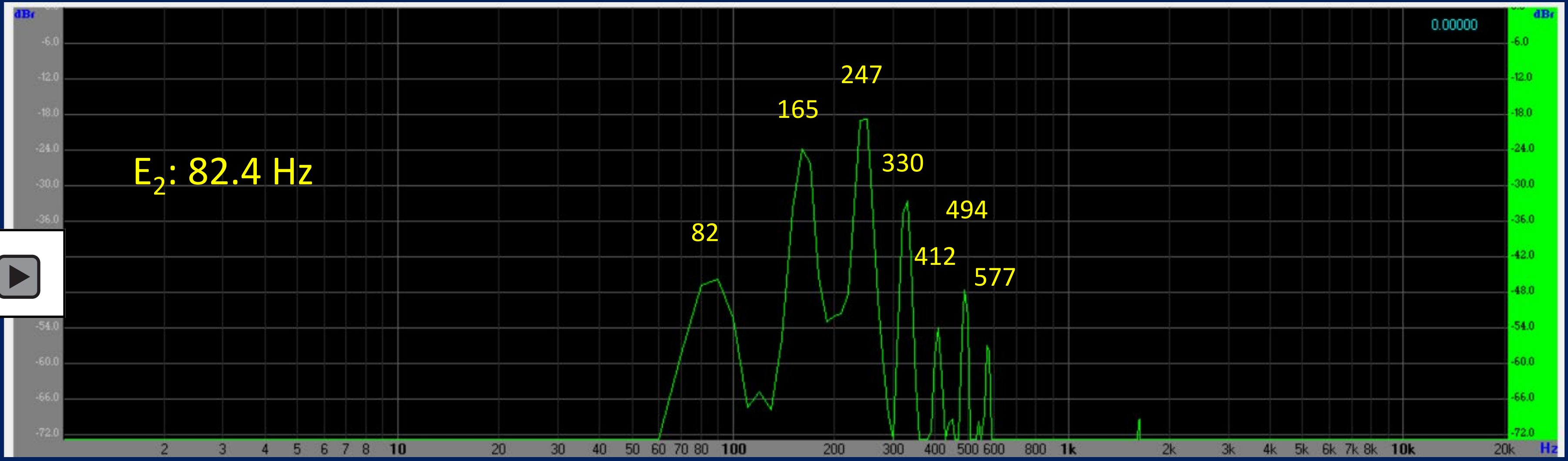
- If dissipation is small, the resonance can be very sharply peaked





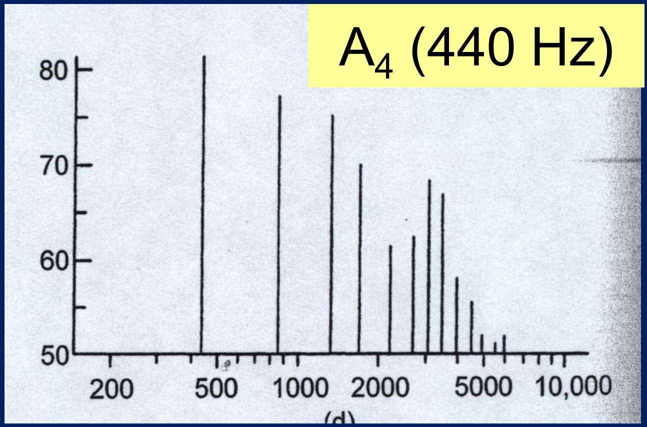
Fourier spectra for acoustic guitar A_2 and E_2 strings

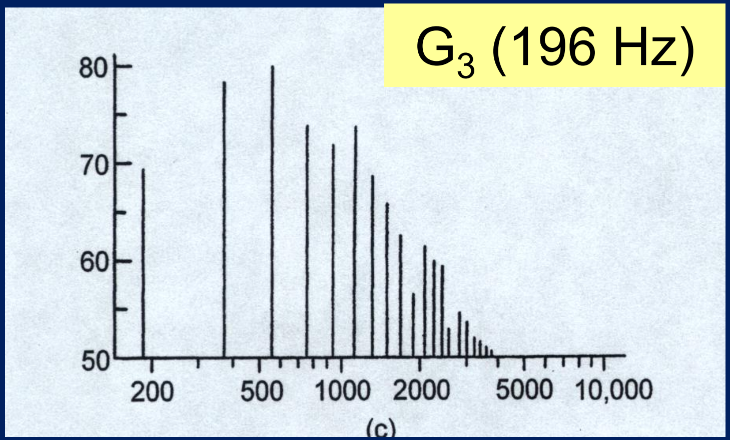


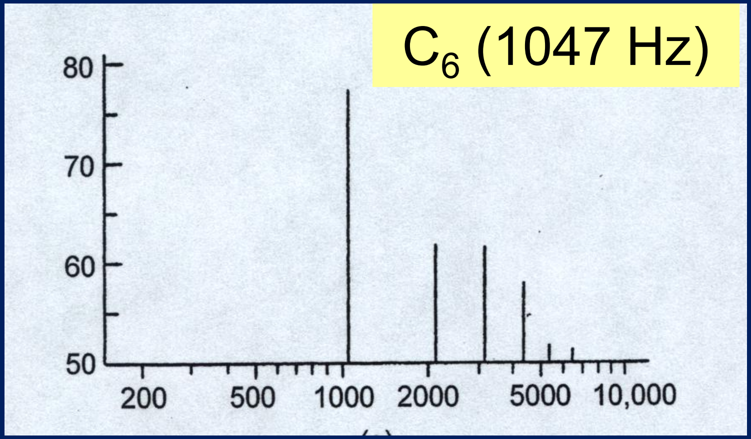


Radiated Violin Sound

- The spectra of different notes can be very complicated
  - The tone quality changes in the different registers of the violin

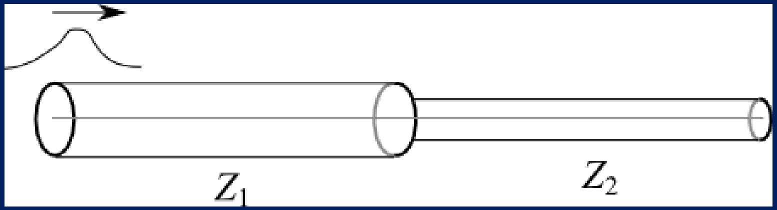






Impedance and Wave Reflections

- Impedance Z is a generalized form of resistance
- Any time a wave encounters a boundary, some fraction R of it may get reflected



$$R = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

- If $Z_1 = Z_2$ (matched impedances)
  - Then $R = 0$ $\Rightarrow$ no reflection
- If Z_1 is much larger than Z_2 or vice versa (very different impedances)
  - Then $R \approx 1$ $\Rightarrow$ almost perfect reflection

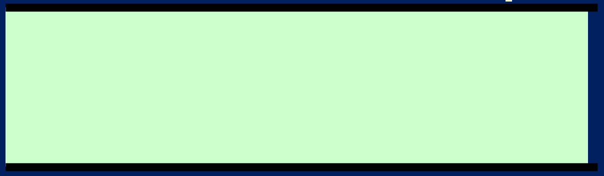
Impedance of Air Pipes

- The acoustic impedance is

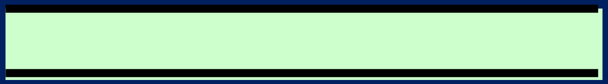
$$Z = \frac{413\ \mathrm{Pa \cdot s/m}}{S}$$

Cross-sectional area

- Inversely proportional to the area of the pipe
  - Wide pipes have a low impedance

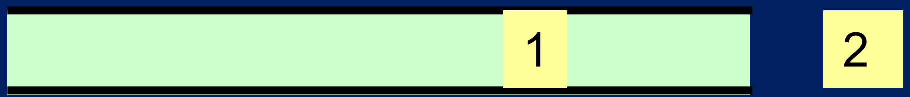


  - Skinny pipes have a high impedance



Trapping Energy in a Pipe

- Consider the impedance of a pipe open on both ends



Area of 2 ≈ infinite
$Z_2 \propto 1/\text{Area} \approx 0$

$$R = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2} \right)^2$$

- $Z_2 = 0$ gives $R = 1$

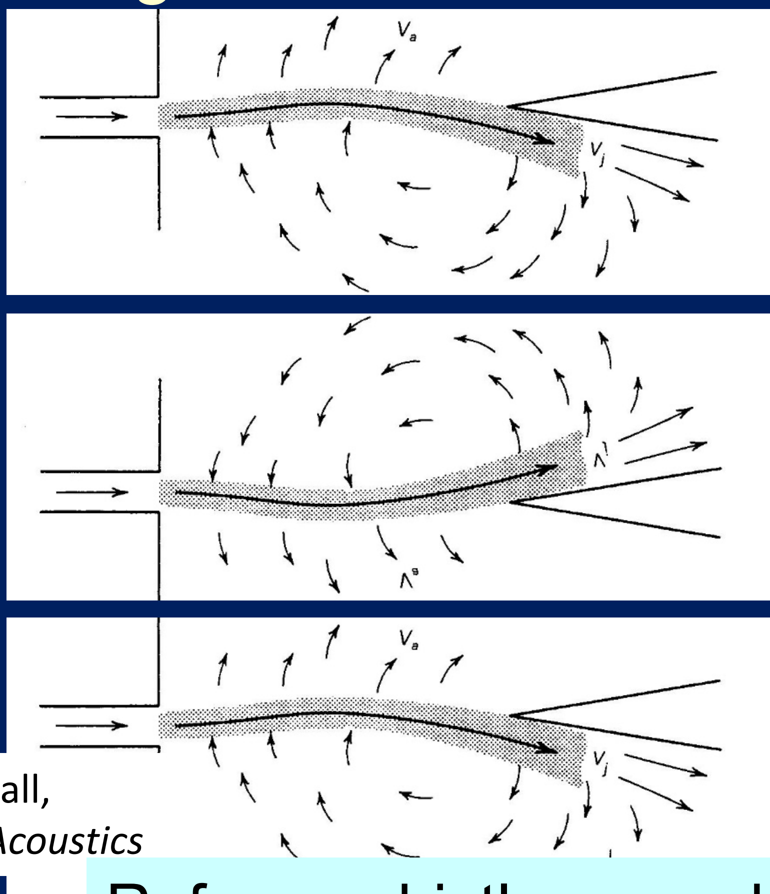
Fraction of energy reflected ≈ 1!

- So even open ended pipes will trap energy!
  - Big area change at ends, so big impedance change

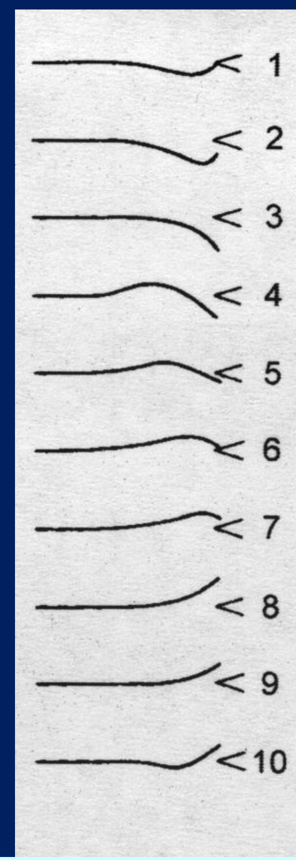
Air columns with fixed lengths are like strings: they have standing waves

Air Jets and Edgetones

- Under the right conditions, an air jet hitting an edge becomes unstable and starts oscillating



Donald Hall, *Musical Acoustics*

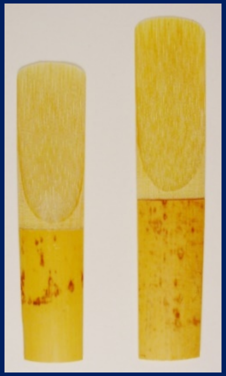


This excites a tone, and is half of what we need to make an instrument

The other half is a resonator to shape and "amplify" the sound

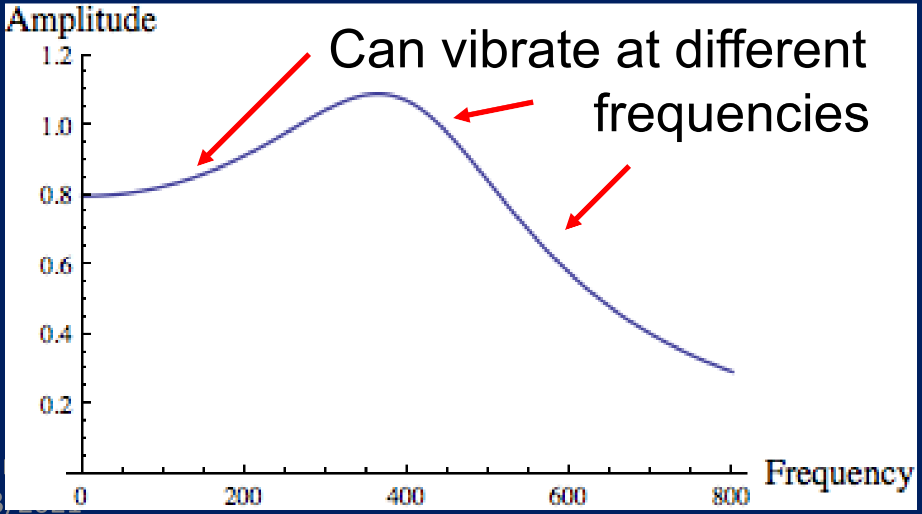
Referee whistle, recorder, organ flue pipes, flute

Vibrating Reeds



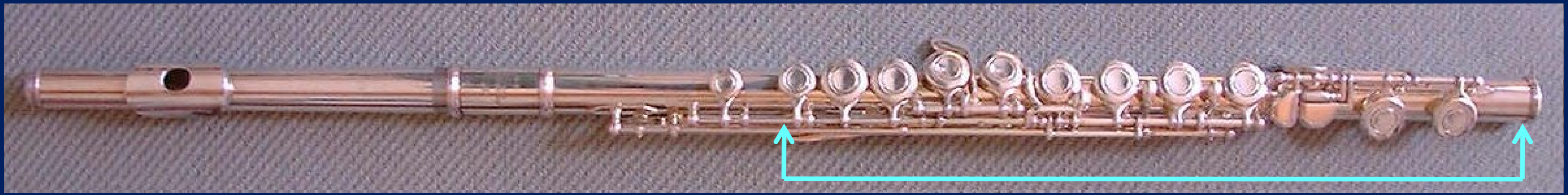
- Cane reeds on woodwinds are soft
- Soft reeds do not vibrate much on their own
  - Vibrational energy is almost immediately dissipated
  - When coupled to an air pipe, soft reeds will vibrate at the natural frequencies *determined by the pipe*
- One soft reed can be used to make a many-note instrument

Vibration response of a soft reed



Clarinet and Flute Tone Holes

- The flute only needs to play one octave in its lowest register

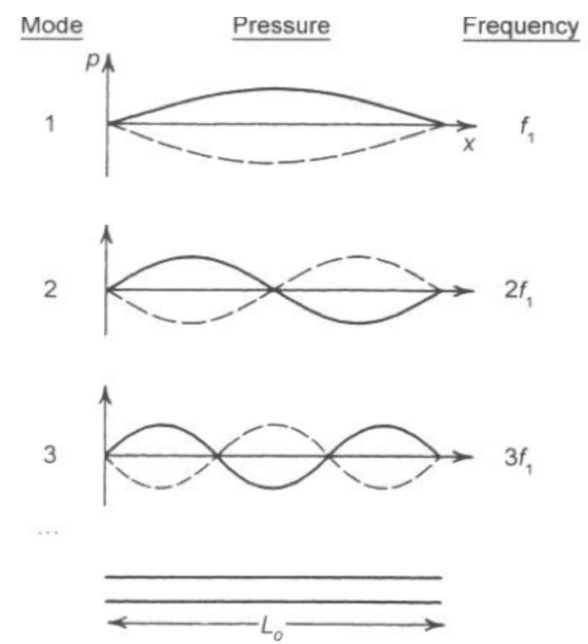
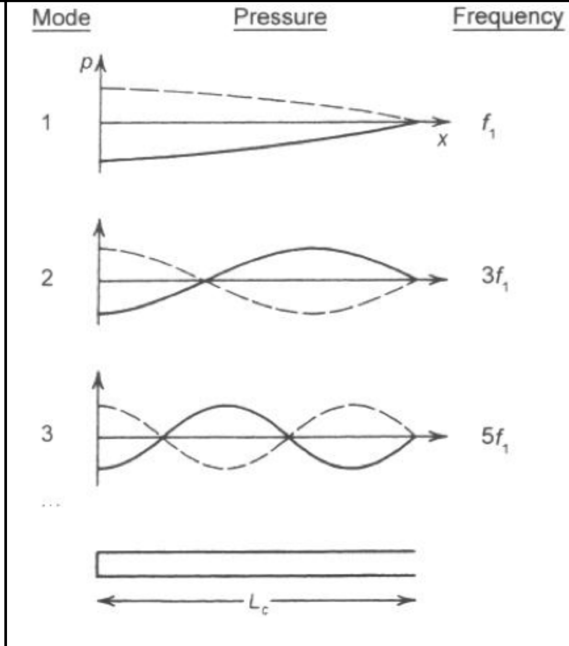
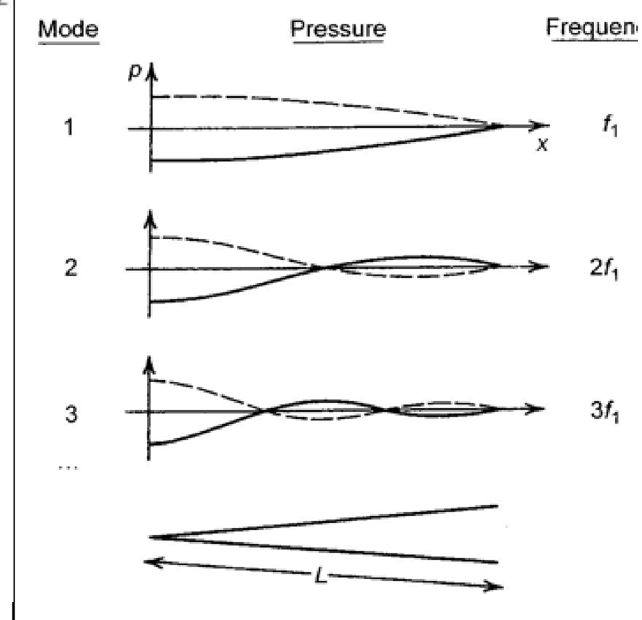


Tone holes in the right half

- The clarinet needs to play a twelfth in its lowest register

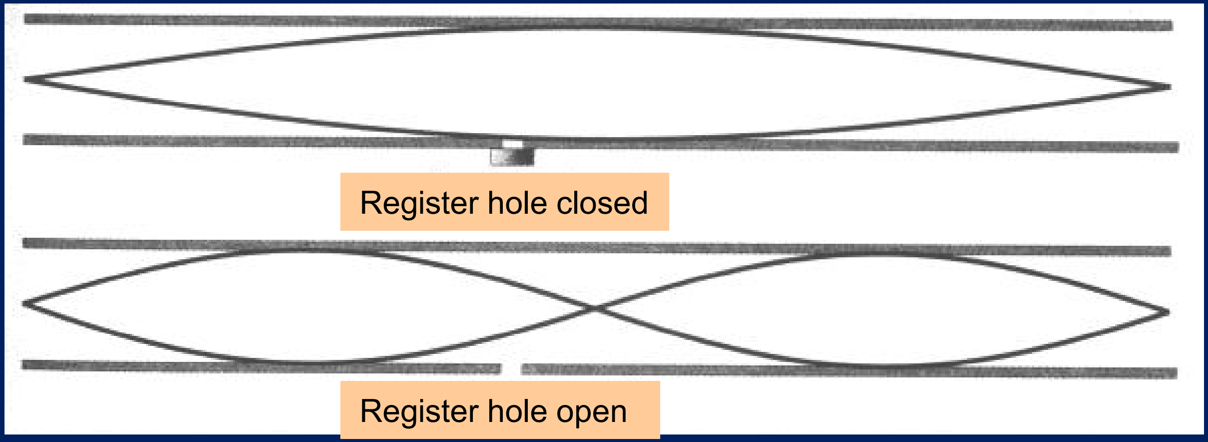
Tone holes in the lower two-thirds

Summary of Pipe Modes

Open cylinder	Closed cylinder	Cone
		
$f_1 = v/2L$	$f_1 = v/4L$	$f_1 = v/2L$
$f_1, 2f_1, 3f_1, 4f_{1,\ldots}$	$f_1, 3f_1, 5f_1, 7f_{1,\ldots}$	$f_1, 2f_1, 3f_1, 4f_{1,\ldots}$

Register Holes

- Register holes are designed to kill the fundamental and allow one of the harmonics to survive
  - Placed near the node of the harmonic

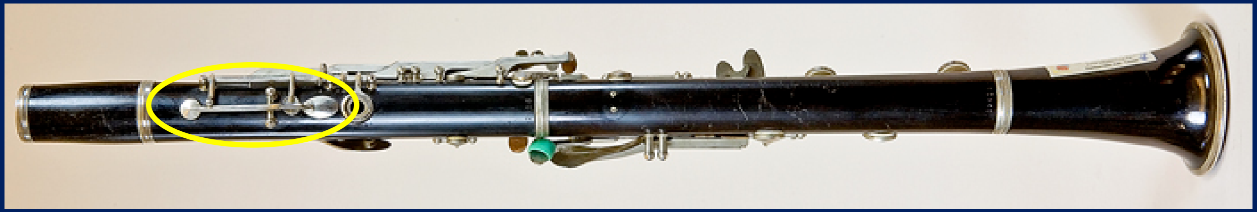


pressure standing waves

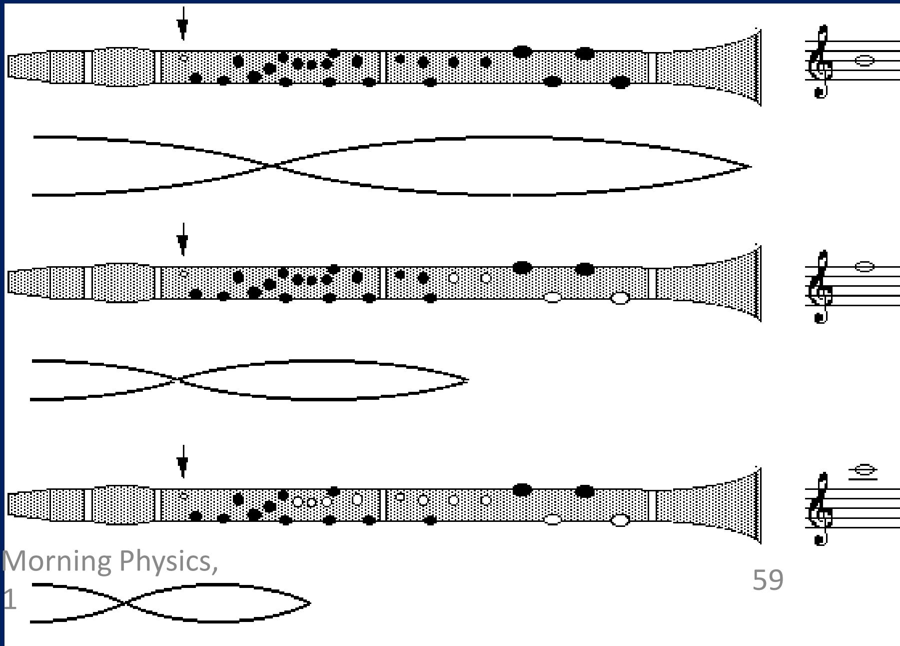
- A register hole e.g. near the middle of an open pipe encourages the second harmonic
  - Open pipe: 2nd mode is an octave
- Improves the tone and makes overblowing more reliable
- Location of the register hole is a compromise for playing a range of notes

Clarinet Register Holes

- The clarinet uses a register key to kill the fundamental when playing in the clarion register

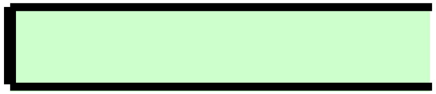


- The location of the register hole is a compromise for playing a range of notes
  - In this example, ideal for F_5, but not as good for lower and higher notes



Wavelength and Frequency

- The *wavelength* is determined by the length and open or closed nature of the pipe
  - How many wavelengths can we fit inside a pipe?

Air pipe	Wavelength	Frequency	Harmonic series
	$\lambda_n = \frac{2L}{n}$	$f_n = \frac{nv}{2L}$	$f_1, 2f_1, 3f_1, 4f_1, \ldots$
	$\lambda_n = \frac{4L}{2n-1}$	$f_n = \frac{(2n-1)v}{4L}$	$f_1, 3f_1, 5f_1, 7f_1, \ldots$

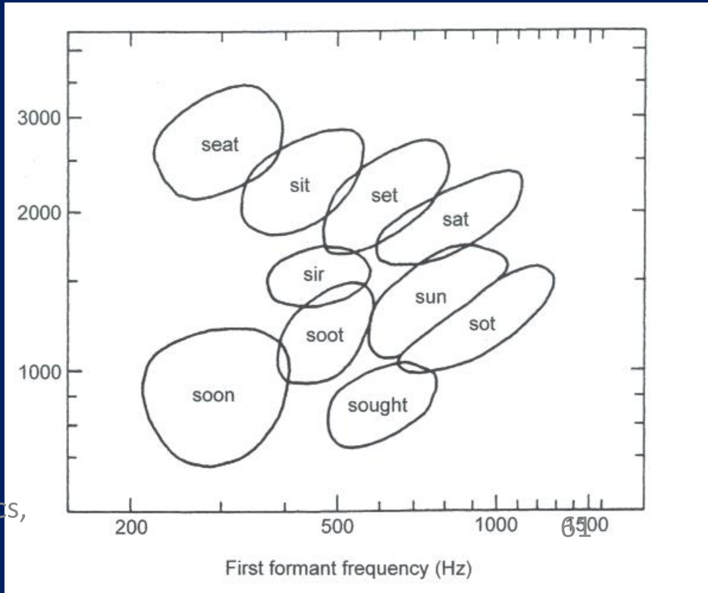
- The *frequency* instead is determined by the speed of sound and the relation $v = \lambda f$

Frequency will depend on temperature!

v = 344 m/s at room temperature

The Human Voice

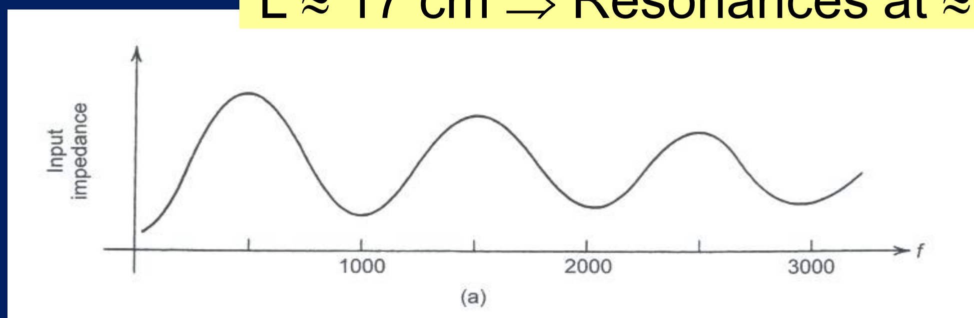
- Basic units of sound: phonemes
  - Consonants: not periodic, no definite pitch
  - Vowels: steady sounds with definite pitch
- Vowels are produced by
  - Exciter: the vocal folds (vocal cords) produce lots of harmonics
  - Resonator: the vocal tract acts as a filter to enhance certain frequencies (*formants*)
- Vowels can generally be characterized by their first two formants



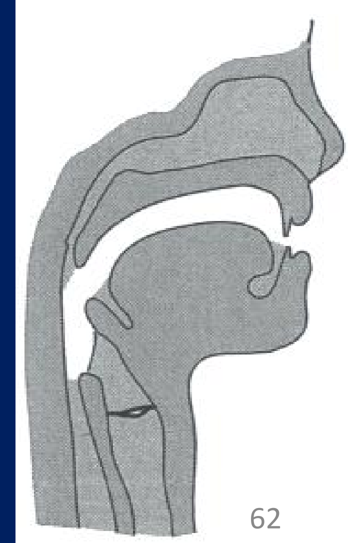
Vocal Tract Resonances

- After the sound is produced by the vocal folds, it is shaped by the vocal tract
- Suppose we model the vocal tract as a 17 cm closed pipe

$L \approx 17$ cm $\Rightarrow$ Resonances at $\approx$ 500 Hz, 1500 Hz, …



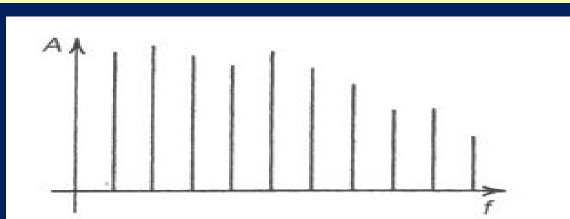
  - Soft walls, so the resonances are very broad
- These natural modes are known as **formants**



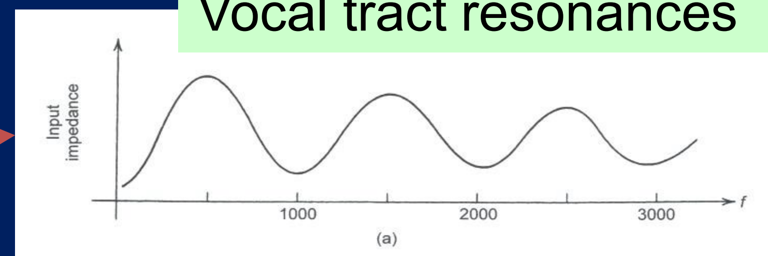
Formants

- The formants act as filter frequencies

Spectrum from voice box

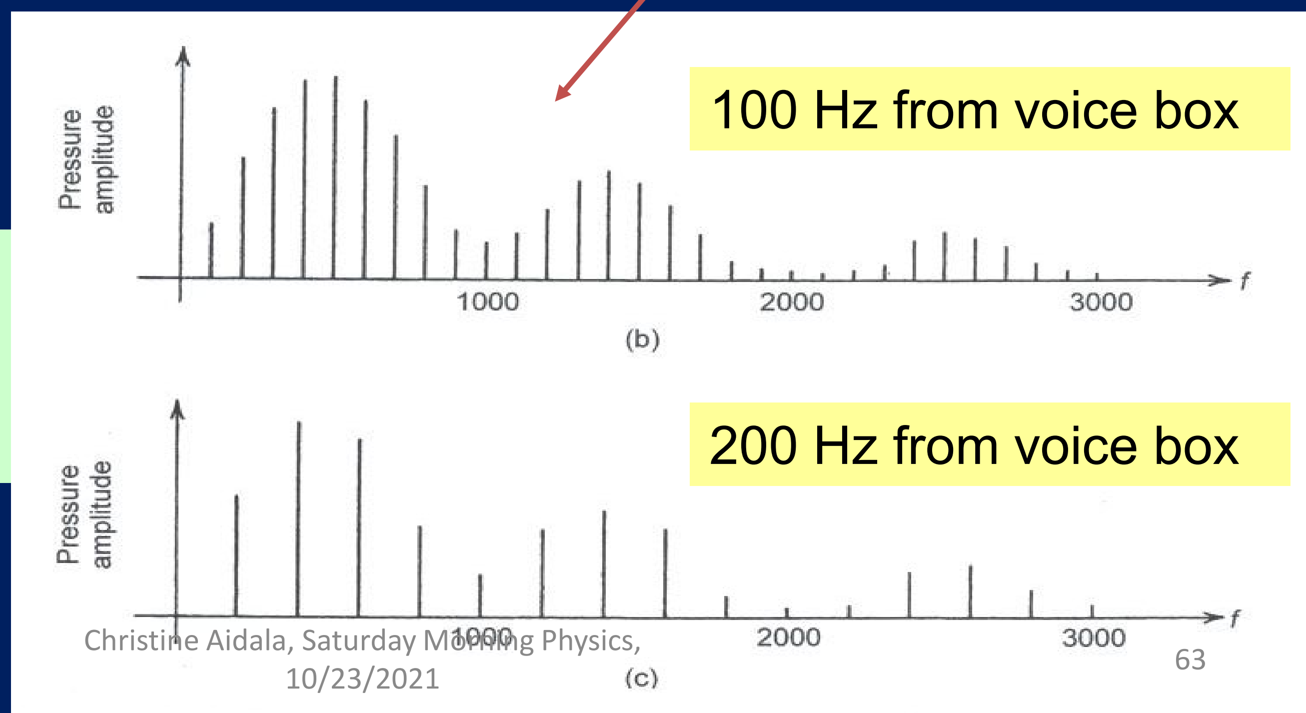


Vocal tract resonances



100 Hz from voice box

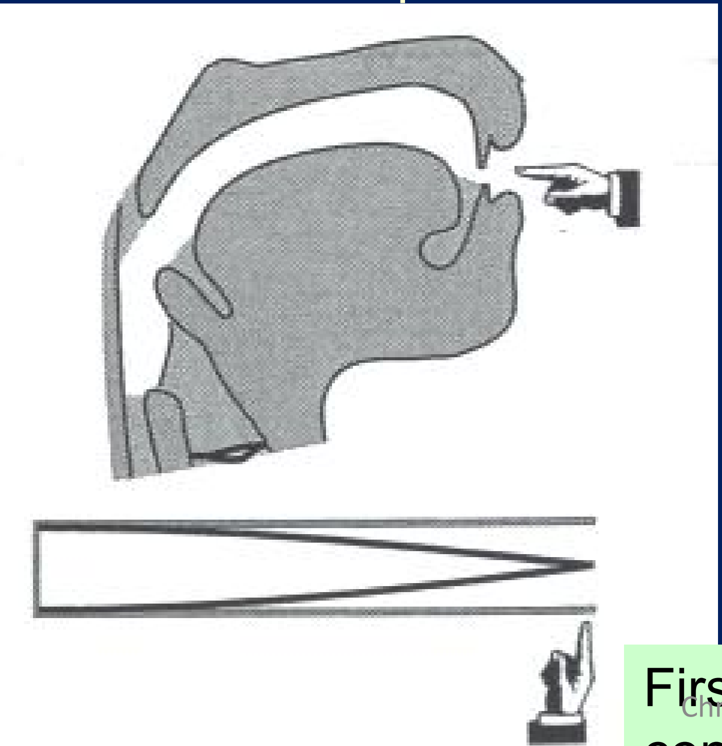
200 Hz from voice box



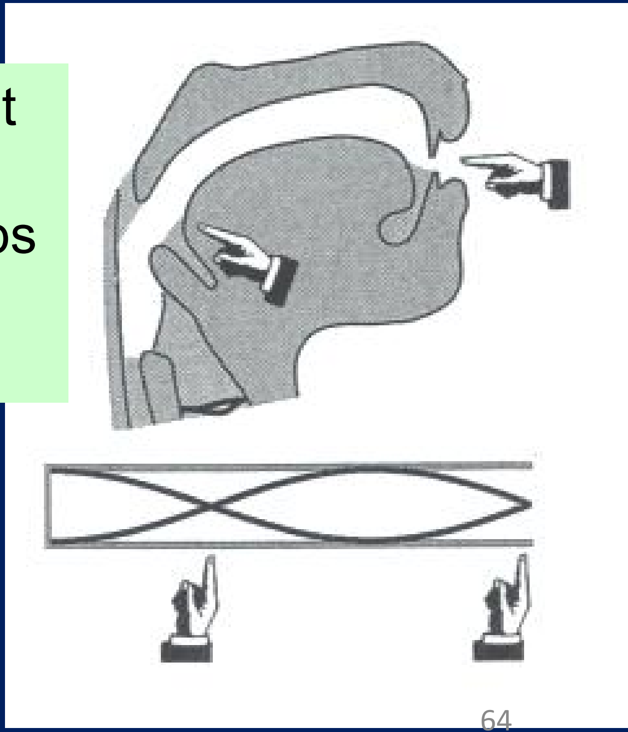
Intensity peaks around formant frequencies

Control of Formant Frequencies

- Our vocal tract is not a perfect cylinder
  - We can control format frequencies by changing the shape of our vocal tract
- Strongest effect at pressure nodes
  - Displacement antinodes

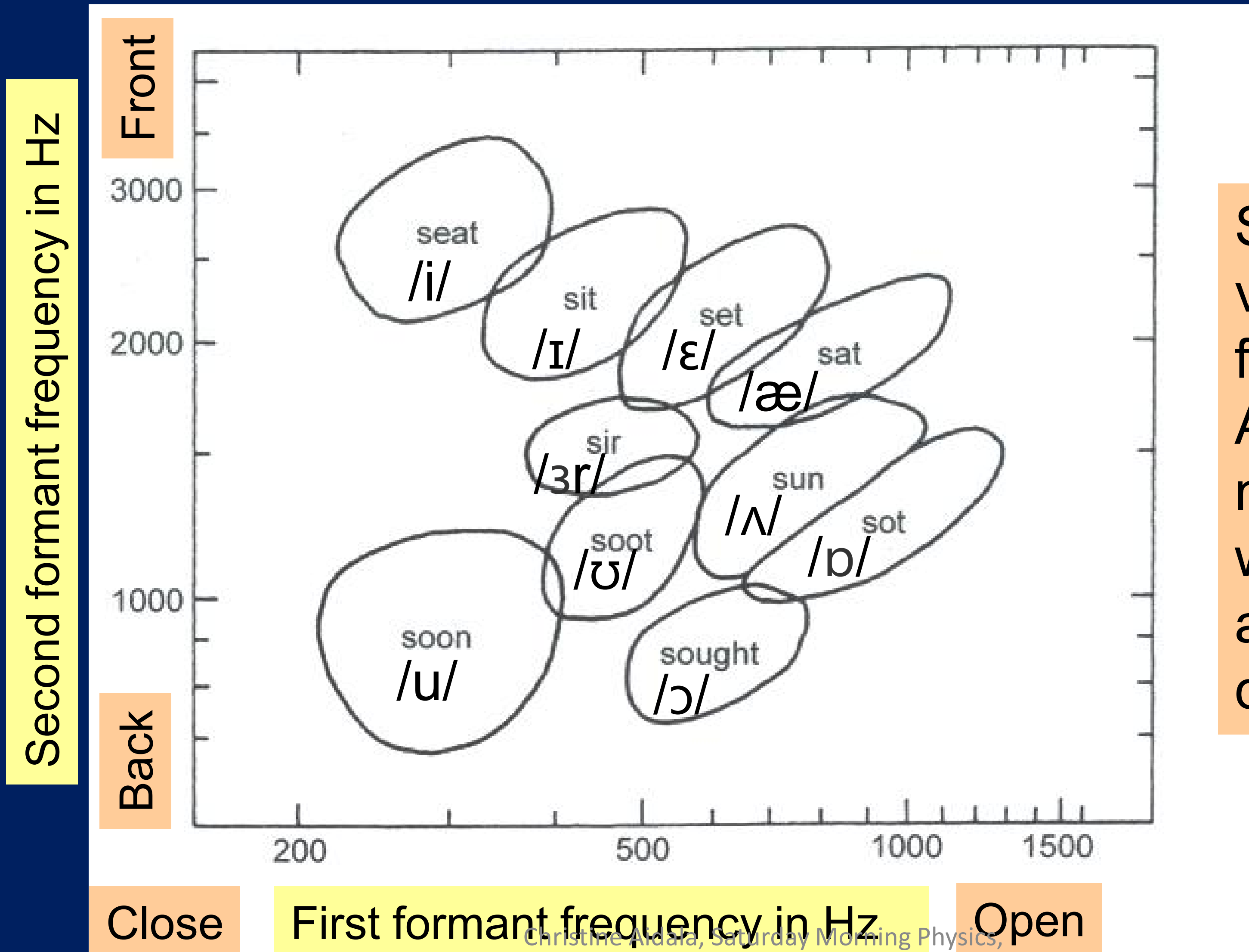


Second formant frequency controlled by lips and back of tongue



First formant frequency controlled by lips

Formant Chart



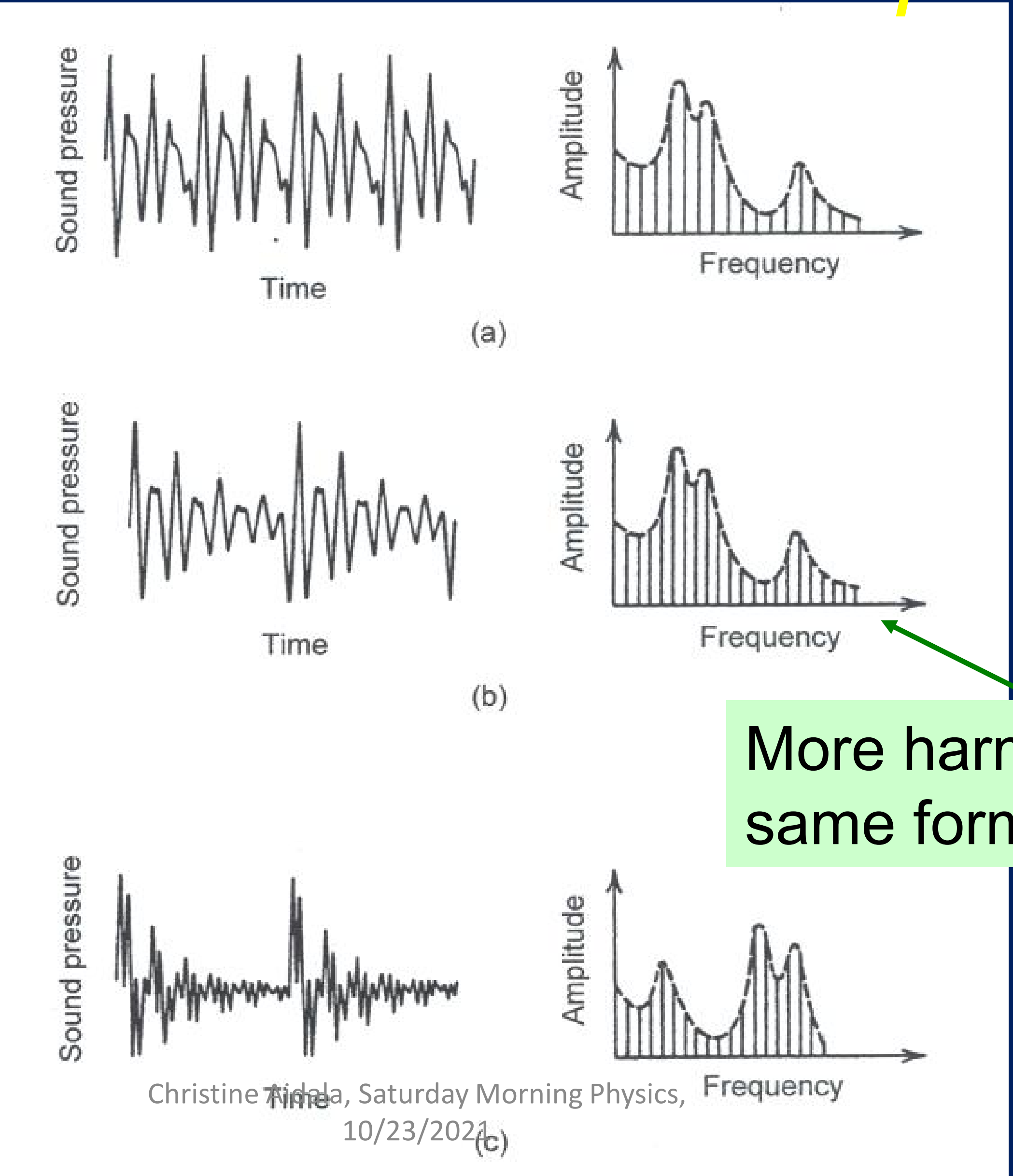
Some variation in formants.
Also between men and women,
and different dialects

Example Waveforms and Spectra

ah at 150 Hz

ah at 90 Hz

uh at 90 Hz



More harmonics, but same formants

Percussion Natural Mode Frequencies

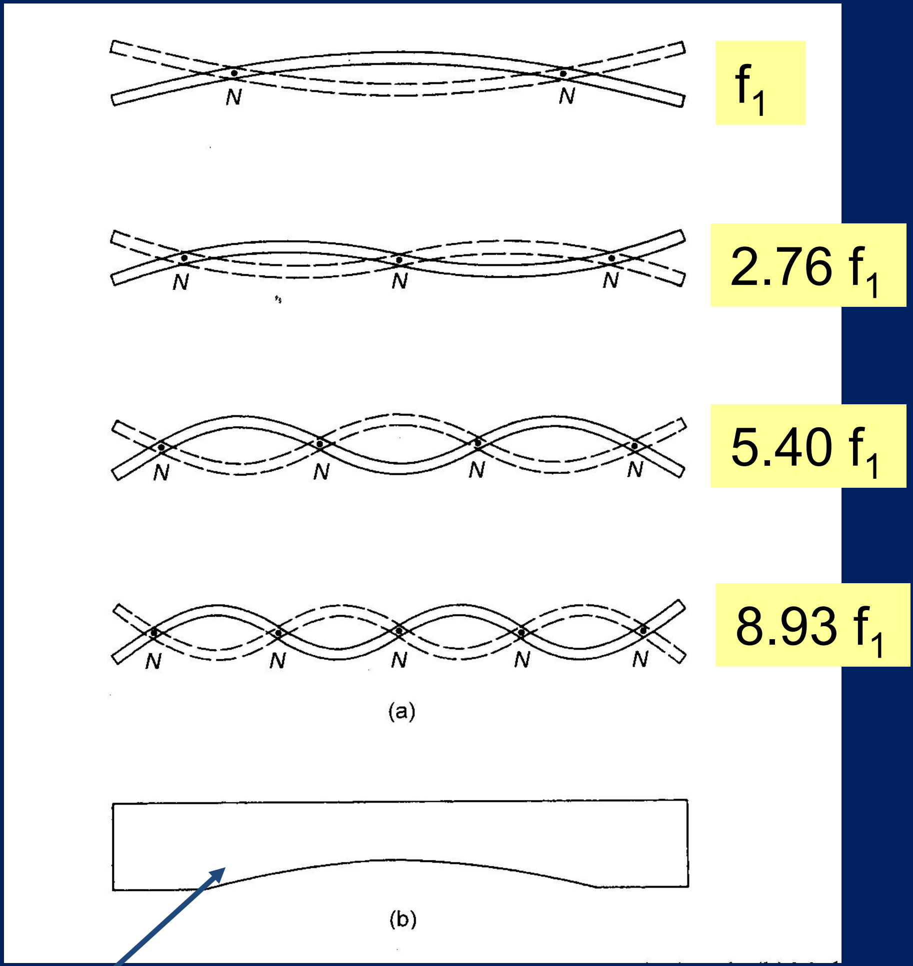
- Since we hear frequencies, we would like to have some idea how the natural mode frequencies are obtained
  - Unfortunately no simple pattern, and depends on many factors
  - Not a harmonic series! ~~f, 2f, 3f, 4f, 5f, 6f,…~~
- General behavior for vibrations

$$f \propto \sqrt{\frac{\text{Restoring force}}{\text{Inertia}}} \propto \sqrt{\frac{\text{Stiffness}}{\text{Mass}}} \rightarrow f_n \propto \sqrt{\frac{\text{Stiffness for bending in mode } n}{\text{Mass vibrating in mode } n}}$$

For the n^{th} mode

Free Bar

- Modes of a uniform bar which is free at both ends
  - xylophone, marimba, metallophone (metal bars)
- General pattern is similar to a clamped bar
  - But different set of frequencies!
- Still not a harmonic series
  - Bending stiffness depends on the mode



Modification of bar to produce more musically helpful frequency ratios – greater thinning for xylophone and marimba bars

Membrane Mode Frequencies

- The frequencies of vibration of a circular membrane can be mathematically determined
  - The n^{th} zero of the m^{th} "Bessel function" $J_m(z)$
- Fortunately, we don't need the details
  - The fundamental frequency f_1 is determined by properties of the drumhead

$$f_1 \propto \sqrt{\frac{\text{restoring force}}{\text{inertia}}} \propto \sqrt{\frac{T}{\sigma}}$$

Tension

Mass per unit area

  - The frequencies of the higher modes can be tabulated as multiples of f_1

The modes are non-harmonic – not due to stiffness (like for bars) but due to geometry of two dimensions

Membrane Mode Frequencies

- We can tabulate the first 10 natural modes of an ideal stretched membrane (an ideal drumhead)
  - Mode numbers labeled from *lowest to highest frequency*
  - Note the mixed pattern of radial and circular nodes