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Convolutions, Factorial Cumulants and Intermittency

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Abstract

It is demonstrated that if multiplicity distributions convolute as a function of a given variable (e.g. rapidity interval), then that variable also shows a true intermittency effect—apparent absence of scale—with a power-law singularity as the variable approaches zero. The intermittency exponents in this case are $\phi_q^K = q - 1$. A simple test for intermittency is proposed making use of the additive property of factorial cumulants for statistically independent subpopulations.

The discovery by UA5 [1] that multiplicity distributions in restricted intervals of pseudo-rapidity exhibit large fluctuations about the average which favor higher multiplicity; and that these fluctuations increase with decreasing pseudo-rapidity interval, $\delta\eta$, gave rise to much excitement at the time. This observation, together with the 'large' bin-by-bin fluctuations or 'spikes' on individual event pseudorapidity distributions from Si+AgBr interactions in cosmic rays at very high energies [2] and in hadron collisions [3] at $\sqrt{s}$=22 GeV, led to the suggestion by Van Hove [3] that "such spectacular spikes might be the result of the formation in the primary hadron-hadron collision of a hot-spot of matter, possibly in the quark-gluon phase." The term 'intermittency' was introduced into particle physics by Bialas and Peschanski [4] to try to explain these unusually large particle density fluctuations or 'spikes' in rapidity as evidence of fractal or intermittent behavior, in analogy to turbulence in fluid dynamics which is characterized by self-similar fluctuations at all scales—the absence of a well defined scale of length [5].

Bialas and Peschanski [4] proposed a formalism to to study non-statistical (more precisely, non-Poisson) fluctuations as a function of the size of rapidity interval by the use of normalized factorial moments of order q :

$$F_q(\delta\eta) = \frac{< n(n-1)\ldots(n-q+1) >}{< n >^q} \quad , \tag{1}$$

where n is the multiplicity in a pseudorapidity interval (bin) of size $\delta\eta$ on a given event and the $< \ >$ brackets indicate averaging over all events. The normalized factorial moments suppress Poisson statistical fluctuations, since all F_q are equal to unity for a Poisson distribution, while the moments of order q enhance spikes since they are non-zero only for events with q or more particles in the interval $\delta\eta$. Intermittency would be indicated by a power-law increase of multiplicity distribution moments over pseudorapidity bins as the bin size is reduced:

$$F_q(\delta\eta) \propto (\delta\eta)^{-\phi_q} \quad . \tag{2}$$

The intermittency formalism stimulated considerable interest and activity for several reasons. It was relatively straightforward [6,4,7] for many experiments to apply the formalism to their data, leading to the observation [8] of the predicted power law behavior in the region $1 \geq \delta\eta \geq 0.1$. The scale-invariant power-law dependence with singular behavior as $\delta\eta \to 0$ was suggestive of the physics of phase transitions [9], chaos [10], and fractals [5,11]. The 'intermittency exponents' ϕ_q determined from Eq. 2 were thought to be related [12–15] to "anomalous fractal dimensions" of the multiplicity distribution which describe how the distribution changes with finer resolution.

Lost in the exhilaration of the era was the fact that multiplicity distributions were well known to be non-Poisson. This had been predicted in 1971 by Mueller [16], who emphasized that the distribution of multiplicity for multiple particle production would not be Poisson unless the particles were emitted independently, without any correlation, but that short range rapidity correlations were expected as a consequence of "Regge-Pole-dominated" reactions [16]. The non-Poisson distribution of multiplicity and short range rapidity correlations were observed and well documented [17].

Mueller [16] introduced a series of moments and correlation functions to describe multiparticle correlations. The Mueller moments, $f_q \equiv <n>^q K_q$, are the factorial cumulants [16,18,19], which are zero if there is no direct q particle correlation; thus, all $f_q, K_q = 0$ for a Poisson. The normalized factorial cumulants, K_q, are just the normalized factorial moments, F_q, with all q-fold combinations of lower order correlations subtracted:

$$\begin{aligned} K_2 &= F_2 - 1 \\ K_3 &= F_3 - (1 + 3K_2) \\ K_4 &= F_4 - (1 + 6K_2 + 3K_2^2 + 4K_3) \\ &\cdots \quad . \end{aligned} \tag{3}$$

The normalized factorial cumulants K_q have one unfortunate computational property for small samples compared to the normalized factorial moments F_q. The $F_q = 0$ for all $q > n_0$, where n_0 is the maximum number of particles observed on an interval $\delta\eta$ in the experiment. The K_q do not have such a simple property and are generally different from zero for all q; they can even change sign [20]. For instance, if an experiment never observes more than 1 particle in a bin, then $F_1 = 1$, $F_{q>1} = 0$, and K_q make wild oscillations [18] $K_q = (-1)^{(q-1)}(q-1)!$.

The unnormalized factorial moments on an interval $\delta\eta$ are the integrals of the q-particle rapidity densities, $\rho_q(y_1, \ldots y_q)$,

$$\int_{\delta\eta} dy_1 \cdots dy_q \; \rho_q(y_1, \ldots y_q) = < n(n-1) \ldots (n-q+1) > = <n>^q \; F_q \quad , \tag{4}$$

while the Mueller moments are the integrals of the Mueller correlation functions:

$$\int_{\delta\eta} dy_1 \cdots dy_q \; C_q(y_1, \ldots y_q) = f_q = <n>^q \; K_q \quad . \tag{5}$$

The Mueller correlation functions C_q can be read from Eqs. 3–5 and are just the q-particle rapidity densities, ρ_q, with all q-fold combinations of lower order densities subtracted. The most straightforward Mueller correlation function is:

$$C_2(y_1, y_2) = \rho_2(y_1, y_2) - \rho_1(y_1)\rho_1(y_2) \quad , \tag{6}$$

which evidently vanishes for the case of no correlation where $\rho_2(y_1, y_2) = \rho_1(y_1)\rho_1(y_2)$.

Mueller's original work [16] emphasized the superiority of factorial cumulants over factorial moments because factorial moments of order q include all q-fold combinations of lower order correlations, which would likely dominate over the direct q-particle correlation:

$$F_q = 1 + \frac{q(q-1)}{2} K_2 + \mathcal{O}K_2^2 + \cdots + K_q \quad . \tag{7}$$

In fact, one of the most interesting obervations [21–24] concerning the systematics of 'intermittency exponents', the moments scaling rule [4]:

$$\phi_q \;=\; \frac{q(q-1)}{2}\phi_2 \qquad , \tag{8}$$

is simply a consequence of the fact [30,25,26] that the 2-particle correlation term K_2 dominates in Eq. 7 for multiple particle production.

The importance of the two-particle correlation to completely determine the multiplicity distribution was pointed out by Fowler and Weiner [27], and more recently by Giovannini and Van Hove [28]; and the application of two-particle short range correlations to the 'intermittency' phenomenology was pioneered by Carruthers, Friedlander, Shih and Weiner [29], Capella, Fialkowski and Krzywicki [30], and Carruthers and Sarcevic [31]. The reduced 2-particle correlation is parameterized in an exponential form

$$R(y_1, y_2) = \frac{C_2(y_1, y_2)}{\rho_1(y_1)\rho_1(y_2)} = \frac{\rho_2(y_1, y_2)}{\rho_1(y_1)\rho_1(y_2)} - 1 = R(0,0)\, e^{-|y_1 - y_2|/\xi} \tag{9}$$

where $\rho_1(y)$, the inclusive single particle density, is assumed constant, and ξ is the correlation length. Then, the integral can be performed on an interval of full width $\delta\eta$, $0 \leq y_1 \leq \delta\eta$, $0 \leq y_2 \leq \delta\eta$:

$$K_2(\delta\eta) = F_2 - 1 = \frac{\int^{\delta\eta} dy_1 dy_2\, C_2(y_1,y_2)}{<n(\delta\eta)>^2} = R(0,0)\, \frac{[1 - \frac{\xi}{\delta\eta}(1 - e^{-\delta\eta/\xi})]}{\delta\eta/2\xi} \quad . \tag{10}$$

The variation of factorial moments with bins size for all the existing data was explained by this formula [29–31], based on conventional short range correlations, without need to resort to singularities or fractal properties. The higher order cumulants were determined from K_2 by the methods of quantum statistics [29], or simply taken as powers [30,31] of K_2, i.e. $K_q = A_q\,(K_2)^{q-1}$, the so called 'linked-pair' approximation, where A_q are constants determined from the data [31]. It was also emphasized that the scaling of the multiplicity distributions, which was not supported by the factorial moment analysis of Eq. 2, might have been masked for the higher order factorial moments by the dominance of K_2 (Eq. 7), so that scaling might be better revealed for higher order correlations by a power law dependence of the normalized factorial cumulants K_q with rapidity interval [31]:

$$K_q(\delta\eta) \propto (\delta\eta)^{-\phi_q^K} \quad . \tag{11}$$

While the intermittency formalism based on moments was particularly well suited to experiments with limited numbers of events [6], experiments with large numbers of events were able to directly determine the distribution of multiplicity and its evolution with pseudorapidity interval. The UA5 collaboration [32,33] made the discovery that the Negative Binomial Distribution (which had been used sporadically for the total multiplicity [34]) gave a "remarkable" description of their measured multiplicity distributions in intervals of rapidity which are not significantly constrained by conservation laws [35,27,36,37], and also of the total multiplicity distribution. A related function, the Gamma Distribution, was found to describe E_T distributions [38].

The Negative Binomial Distribution (NBD) has an additional parameter k compared to a Poisson distribution, and becomes Poisson in the limit $k \to \infty$ and Binomial for k equal to a negative integer (hence the name). The normalized factorial moments (F_q) and cumulants (K_q) of the NBD are only slightly more complicated than those of a Poisson [18,19]:

$$F_q = F_{(q-1)}(1 + \frac{q-1}{k}) \qquad\qquad K_q = \frac{(q-1)!}{k^{q-1}} = (q-1)!\;(K_2)^{q-1} \quad . \tag{12}$$

For large values of k, the normalized factorial cumulants go to zero as a power of $1/k$; and the NBD parameter in the form $1/k$ is used by statitisticians [39] as a measure of departure from a Poisson law. Clearly, for the NBD, the two-particle short range correlation, which determines $K_2(\delta\eta) = 1/k(\delta\eta)$, also determines the entire distribution [28,40], including its evolution with $\delta\eta$. A noteworthy property of the NBD, shared by the Gamma distribution, concerns convolutions: the probability distribution of the sum of z independent variables, each distributed as an NBD with mean $\mu = <n>$ and parameter k, is the z-fold convolution of the distribution, which is an NBD with mean $z\mu$ and parameter zk, so that the ratio μ/k remains constant for the convolutions. The NBD becomes a Gamma distribution in the limit $\mu \gg k > 1$.

Since the pioneering work of UA5 [32], many other experiments have shown that the NBD provides excellent fits to charged multiplicity distributions in restricted $\delta\eta$ intervals for all reactions studied, for example: p+p (NA22 [41]), $e^+ + e^-$ (HRS [42]), μ+p DIS (EMC [43]), S+S central (NA35 [44]), O+Cu central (E802 [45]). It is interesting to note that all these measurements show the same effect—linear dependences of the NBD parameter $k(\delta\eta)$ with the pseudo-rapidity interval $\delta\eta$, or equivalently with the mean multiplicity in the interval $\mu = <n(\delta\eta)>$, with non-zero intercept, $k(0) \neq 0$ (see Fig. 1).

By rewriting Eq. 10 for the case of the NBD, where $k = 1/K_2$:

$$k(\delta\eta) = \frac{1}{K_2(\delta\eta)} = \frac{<n(\delta\eta)>^2}{\int^{\delta\eta} dy_1 dy_2\, C_2(y_1, y_2)} = \frac{1}{R(0,0)} \frac{\delta\eta/2\xi}{[1 - \frac{\xi}{\delta\eta}(1 - e^{-\delta\eta/\xi})]} \quad , \qquad (13)$$

it is easy to see that the non-zero intercept $k(0) \neq 0$ implies both that the correlation length is finite, $\xi \neq 0$, and that there is no intermittency. For finite $\xi > 0$, the $\delta\eta$ interval can become smaller than the correlation length, so that $\delta\eta \ll \xi$ as $\delta\eta \to 0$, in which case the square bracket $\to \delta\eta/(2\xi)$, so that $k(\delta\eta) \to 1/R(0,0) \neq 0$ is constant as $\delta\eta \to 0$. Thus, for the NBD, $K_q(0)$ is finite for all orders in this limit so there is no intermittency [46].

The two-particle short range correlation length ξ and strength $R(0,0)$ can be derived from fits of Eq. 13 to the measured $k(\delta\eta)$, as shown on Fig. 1. The fit parameters reproduce the directly measured correlations very well for the hadron collisions [17], $\xi \sim 1-2$ units of rapidity and $R(0,0) \sim 0.6$. However, the fitted parameters for the heavy ion collisions [45], $R(0,0) = 0.031 \pm 0.005$, $\xi = 0.18 \pm 0.05$, are considerably smaller than those for hadron collisions but are still clearly finite and measurable. The weakened and very short range rapidity correlations in collisions of relativistic heavy ions had been predicted several years ago [29,30,47,48]—in nucleus-nucleus collisions, the conventional hadron short-range correlations should be washed out by the random superposition of correlated sources [30,13,26], so that only the quantum-statistical Bose-Einstein (B-E) correlations of identical particles will remain [29,30,25]. As the Bose-Einstein effect represents a very short range correlation in the invariant difference in four-momenta ($Q = p_1 - p_2$) of the two identical particles, it should come to dominate the conventional short-range correlation at very small intervals and should produce dramatic differences for the case of identical or non-identical particles—even in hadron collisions—especially for intermittency analyses which study small volumes in multi-dimensional phase space. In fact, the relationship of intermittency and B-E correlations [49,51,50,52] has been convincingly demonstrated. However, for the purposes of the present discussion, the B-E correlation gives a finite correlation length related to the radius of the interaction volume [53] and thus does not result in true intermittency. It is interesting to consider what would happen if the correlation length would actually go to zero.

In the limit of very small correlation length, $\xi \to 0$ (i.e. $\xi \ll \delta\eta_0$, where $\delta\eta_0$ is the smallest region that can be resolved), Eq. 13 takes the form of the limit $\delta\eta \gg \xi$:

$$k(\delta\eta) = \frac{1}{K_2(\delta\eta)} = \frac{\delta\eta}{2\xi\, R(0,0)} \quad , \tag{14}$$

so that $k(\delta\eta)$ is directly proportional to $\delta\eta$, as in the case of convolutions of independent bins [29], and thus $k(0) = 0$. In this limit, $K_2(\delta\eta) = 1/k(\delta\eta)$ is singular in $\delta\eta$, with 'intermittency exponents' $\phi_2^K = 1$; and $\phi_q^K = q - 1$ for the NBD or the 'linked-pair' approximation. This corresponds to the case of true intermittency in multiparticle production, the apparent absence of scale, or, more precisely, a scale much smaller than the resolution of a particular measurement. For $\xi \to 0$, the proportionality constant $2\xi\, R(0,0)$ in Eq. 14 just corresponds to the effect of the two-particle correlation as an integral over $\delta\eta$ intervals too large for the differential form (Eq. 9) to be resolved. Evidently, if the distribution is actually singular, then any interval is too large to resolve the differential form.

The above discussion demonstrates that true intermittency, $\xi \rightarrow 0$, occurs for Negative Binomial Distributions only when the NBD parameter $k(\delta\eta)$ is directly proportional to $\delta\eta$, with a zero intercept, $k(0) = 0$, which is the case for statistical independence in the variable $\delta\eta$ so that the multiplicity distributions convolute as the interval $\delta\eta$ is extended*. The linear dependence is seen in all experiments (see Fig. 1). However, the intercepts $k(0)$ are non-zero to a large statistical significance in all cases, indicating a finite correlation length $\xi \neq 0$ and therefore no intermittency!

The relationship between convolutions, statistical independence and intermittency is valid in general for the case of constant density, $\rho_1(y) = dn/d\eta$: statistical independence in the variable $\delta\eta$ implies a singularity in $K_q(\delta\eta)$ as $\delta\eta \rightarrow 0$ with 'intermitency exponents' $\phi_q^K = q - 1$. This can be demonstrated by the use of the Mueller factorial cumulants, which are additive for statistically independent subpopulations [19]. For any order of $f_q(\delta\eta)$ where there is statistical independence in the variable $\delta\eta$, the additivity for statistically independent subpopulations—for any resolution $\delta\eta$—will guarantee that $f_q(\delta\eta)$ is directly proportional to $\delta\eta$ with a zero intercept, $f_q(0) = 0$. The test for the direct proportionality of $f_q(\delta\eta)$ on $\delta\eta$ is best performed using the ratio $f_q(\delta\eta)/{<}n(\delta\eta){>}$ which should be a constant if the intercept is zero. Obviously, if $f_q(\delta\eta)/{<}n(\delta\eta){>} = B_q \neq 0$, then, as $\delta\eta \rightarrow 0$, $K_q(\delta\eta)$ diverges:

$$K_q(\delta\eta) = \frac{f_q(\delta\eta)}{<n(\delta\eta)>^q} = \frac{B_q}{<n(\delta\eta)>^{q-1}} \propto (\delta\eta)^{-(q-1)} \quad ; \qquad (15)$$

and there is intermittency with $\phi_q^K = q - 1$.

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*The apparent contradiction of statistically independent non-Poisson distributions thus indicates either a singular rapidity correlation or the possibility that non-Poisson statistics could result from correlation in a phase space variable other than (pseudo)rapidity such as: impact paramteter [54], multiple quark/gluon interactions [55], resonances [56], clusters [28,57], etc.

The use of normalized or unnormalized factorial cumulants is not advantageous as a test for weaker forms of intermittency, where the divergence is less strong than the case of statistitical independence, since, for all the data in Fig. 1, a false signal for intermittency is given by a fit to Eq. 11. This is an artifact of fitting the form of Eq. 11 to the form of Eq. 10, which is just the inverse of the plot shown in Fig. 1: $K_2(\delta\eta) = 1/k(\delta\eta)$ increases with decreasing $<n(\delta\eta)>$, which gives a divergent exponent in a power-law fit. However, $K_2(\delta\eta)$ never actually diverges since it becomes a constant, $R(0,0)$, as $<n(\delta\eta)> \rightarrow 0$.

Thus, in order to really find intermittency with a simple test, it is essential to define the 'inverse' quantities $T_q(\delta\eta)$:

$$K_q(\delta\eta) \equiv \frac{(q-1)!}{(T_q(\delta\eta))^{q-1}} \quad . \tag{16}$$

Then, if $T_q(\delta\eta) \propto <n(\delta\eta)>^{\tau_q}$ as $<n(\delta\eta)> \rightarrow 0$: there is no intermittency if $\tau_q \leq 0$; and there is intermittency, with intermittency exponents $\phi_q^K = \tau_q\,(q-1)$, if $\tau_q > 0$. Obviously, if $T_q(0) \neq 0$, there is no intermittency, since $K_q(0)$ is finite.

The new test for intermittency consists of two parts. First, look for statistical independence by making a plot versus $<n(\delta\eta)>$ of the ratio:

$$\frac{1}{\left(\frac{dn}{d\eta}\right)} \frac{f_q(\delta\eta)}{\delta\eta} = \frac{f_q(\delta\eta)}{<n(\delta\eta)>} \quad . \tag{17}$$

If this ratio is a constant $B_q \neq 0$, as a function of $<n(\delta\eta)>$, it is clear evidence of statistical independence and the conditions of Eq. 15. A more likely outcome is that the ratio in Eq. 17 will become constant only for large $<n(\delta\eta)>$, but will approach zero as $<n(\delta\eta)> \rightarrow 0$. The test for intermittency will consist in determining the rate of this approach. For this purpose, it is proposed to examine the quantities $T_q(\delta\eta)$ as a function of $<n(\delta\eta)>$, where

$$T_q(\delta\eta) = \frac{<n(\delta\eta)>}{\left(\frac{1}{(q-1)!} \frac{f_q(\delta\eta)}{<n(\delta\eta)>}\right)^{\frac{1}{q-1}}} = \left(\frac{(q-1)!}{K_q(\delta\eta)}\right)^{\frac{1}{q-1}} \quad . \tag{18}$$

In the case of the NBD, $T_q(\delta\eta) = k(\delta\eta)$, and this plot is identical to Fig. 1 for all orders. In the linked-pair approximation, the plots for the different orders will have identical dependences on $<n(\delta\eta)>$, but the normalizations will be different, from which the factors A_q can be determined. Note that this is designed to be a linear plot on both axes, with a simple test for intermittency: intermittency exists only if the intercept $T_q(0) = 0$; if $T_q(0) \neq 0$ there is no intermittency.

The intermittency exponents are determined by the rate of approach of $T_q(\delta\eta)$ to zero with $< n(\delta\eta) >$. For the case of statistical independence, $f_q(\delta\eta)/ < n(\delta\eta) >$ is constant, again resulting in a direct proportionality of $T_q(\delta\eta)$ on $<n(\delta\eta)>$, with zero intercept, giving the intermittency exponents (Eq. 16), $\phi_q^K = q - 1$, $\tau_q = 1$. In principle, intermittency can occur for any value of $\tau_q > 0$ and thus can be stronger or weaker than the case of statistical independence. However, in the simple model of exponential correlation with linked-pairs, Eq. 13 applies, and the only solution for intermittency is $\xi \to 0$, leading to Eq. 14, with statistical independence and $\tau_q = 1$.

In the case $T_q(0) \neq 0$, which corresponds to no intermittency, Eq. 13 can be used to fit the data to find the effective average correlation length ξ_q and strength $R_q(0,0)$ for each order. Of course, nature may be more complicated, and the simple exponential correlation length (Eq. 9) and/or the 'linked-pair' approximation may not be valid. Nevertheless, a plot of the form of Eq. 18 has the advantage that a correlation length dependence of any form can be easily visualized on a linear scale for each order q, and thus can presumably be determined, whatever the form.

This test is designed to be used in the case of high statistics, with a large number of events. In order to sensibly define $T_q(\delta\eta)$, the normalized fatorial cumulants $K_q(\delta\eta)$ must be positive and statistically different from zero. Evidently, if $K_q(\delta\eta)$ is compatible with zero, the experimental statistics are not adequate for the measurement of a deviation from a Poisson distribution. Similarly, if $F_q(\delta\eta)$ is unmeasured or statistically compatible with zero, the $K_q(\delta\eta)$ for that order and any higher should be ignored. It should be noted that there has been some recent activity on (normalized) factorial cumulants [52,58–63,20], including some warnings of calculational difficulties for low statistics [59,20]. Nevertheless, high-statistics experiments providing NBD fits still give the best evidence and constraints for higher order correlations.

In summary, the connection between statistical independence and intermittency has been demonstrated. A new and simple test for intermittency has been proposed using the Mueller factorial cumulants which are additive for statistical independent subpopulations. In a simple model of two-particle correlations [27,29–31], true intermittency occurs when the correlation length $\xi \to 0$, in which case the normalized factorial cumulants become singular with finer resolution, with 'intermittency exponents' $\phi_q^K = q-1$. Such a dependence had been proposed [14] as a "possible signal of the formation of quark-gluon plasma and its subsequent phase transition into hadrons," whereas in the present case it is simply a consequence of the 'linked-pair' approximation or the NBD. It will be interesting to see how nature resolves this issue.

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FIGURES

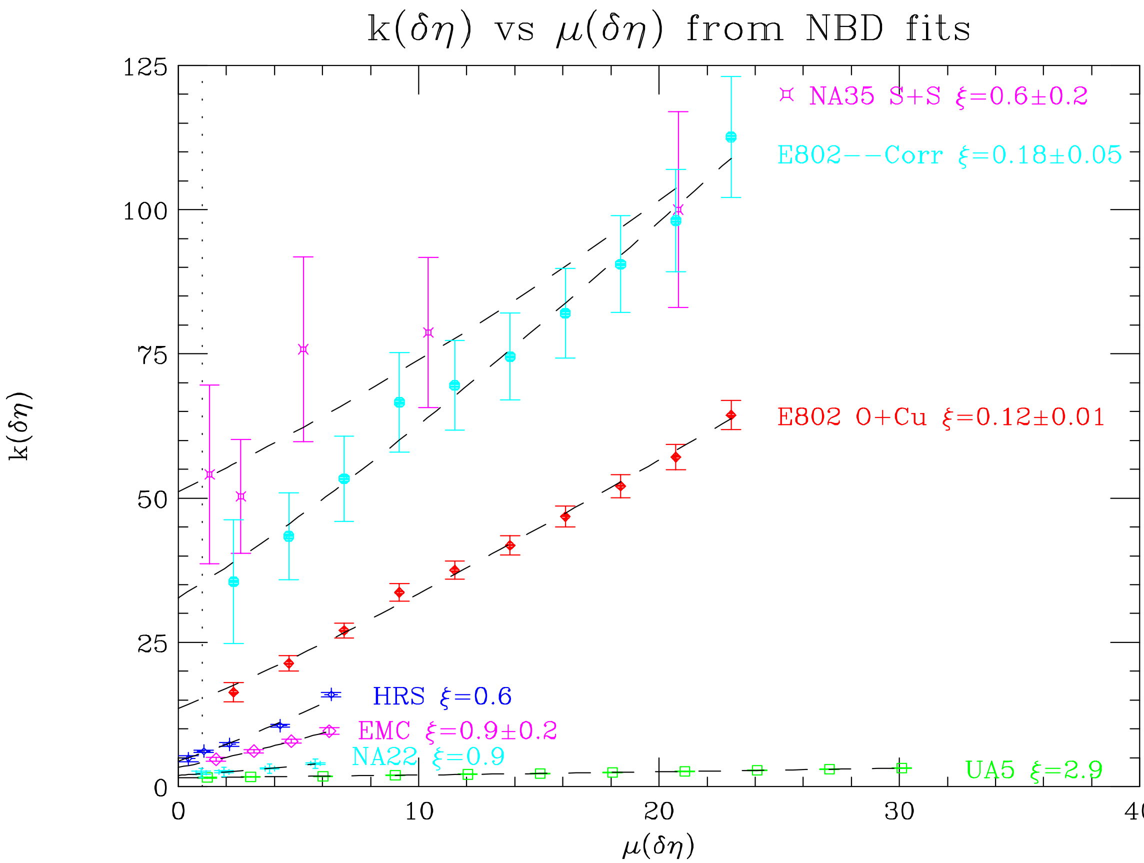

FIG. 1. The NBD parameter $k(\delta\eta)$ or the intermittency test $T_q(\delta\eta)$ as a function of the mean multiplicity in the pseudo-rapidity interval, $\mu = < n(\delta\eta) > = dn/d\eta \times \delta\eta$: UA5 $\bar{\text{p}}$+p $\sqrt{s}$ = 540 GeV ($dn/d\eta$=3.01), NA22 p+p $\sqrt{s}$ = 22 GeV (1.90), EMC μ-p DIS W = 18 − 20 GeV (1.57), HRS $e^+ + e^-$ 2-Jet $\sqrt{s}$ = 29 GeV (2.12), E802 O+Cu central P_{beam} = 14.6A GeV/c (23.0), E802 corrected $k^{\text{C}}(\delta\eta)$, NA35 S+S central E_{beam} = 200A GeV (10.4). The dashed lines are fits to Eq. 13 with the parameters ξ indicated.